13.8 The Divergence Theorem

A Click here for answers.

S Click here for solutions.

1. Verify that the Divergence Theorem is true for the vector field $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + yz\,\mathbf{j} + 3z^2\,\mathbf{k}$ on the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 1$.

2–12 Use the Divergence Theorem to calculate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$; that is, calculate the flux of $\mathbf{F}$ across S.

2. $\mathbf{F}(x, y, z) = 3y^2z^3\,\mathbf{i} + 9x^2yz^2\,\mathbf{j} - 4xy^2\,\mathbf{k}$,
S is the surface of the cube with vertices $(\pm 1, \pm 1, \pm 1)$

3. $\mathbf{F}(x, y, z) = x^2y\,\mathbf{i} - x^2z\,\mathbf{j} + z^2y\,\mathbf{k}$,
S is the surface of the rectangular box bounded by the planes $x = 0$, $x = 3$, $y = 0$, $y = 2$, $z = 0$, and $z = 1$

4. $\mathbf{F}(x, y, z) = -xz\,\mathbf{i} - yz\,\mathbf{j} + z^2\,\mathbf{k}$,
S is the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$

5. $\mathbf{F}(x, y, z) = 3xy\,\mathbf{i} + y^2\,\mathbf{j} - x^2y^4\,\mathbf{k}$,
S is the surface of the tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$

6. $\mathbf{F}(x, y, z) = x^3\,\mathbf{i} + y^3\,\mathbf{j} + z^3\,\mathbf{k}$,
S is the sphere $x^2 + y^2 + z^2 = 1$

7. $\mathbf{F}(x, y, z) = x^3\,\mathbf{i} + 2xz^2\,\mathbf{j} + 3y^2z\,\mathbf{k}$,
S is the surface of the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane

8. $\mathbf{F}(x, y, z) = ye^{z^2}\,\mathbf{i} + y^2\,\mathbf{j} + e^{xy}\,\mathbf{k}$,
S is the surface of the solid bounded by the cylinder $x^2 + y^2 = 9$ and the planes $z = 0$ and $z = y - 3$

9. $\mathbf{F}(x, y, z) = z\cos y\,\mathbf{i} + x\sin z\,\mathbf{j} + xz\,\mathbf{k}$,
S is the surface of the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $2x + y + z = 2$

10. $\mathbf{F}(x, y, z) = (x + e^{y\tan z})\,\mathbf{i} + 3xe^{xz}\,\mathbf{j} + (\cos y - z)\,\mathbf{k}$,
S is the surface with equation $x^4 + y^4 + z^4 = 1$

11. $\mathbf{F}(x, y, z) = xy^2\,\mathbf{i} + yz\,\mathbf{j} + zx^2\,\mathbf{k}$,
S is the surface of the solid that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ and between the planes $z = 1$ and $z = 3$

12. $\mathbf{F}(x, y, z) = (x^3 + yz)\,\mathbf{i} + x^2y\,\mathbf{j} + xy^2\,\mathbf{k}$,
S is the surface of the solid bounded by the spheres $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 9$

Answers

E Click here for exercises.

S Click here for solutions.

2. 8

3. 24

4. 0

5. $\frac{5}{24}$

6. $\frac{12}{5}\pi$

7. 32π

8. $-\frac{81}{2}\pi$

9. $\frac{1}{6}$

10. 0

11. 27π

12. $\frac{3376}{15}\pi$

Solutions

E Click here for exercises.

A Click here for answers.

1. div $\mathbf{F} = 8z$, so

$\iiint_E \operatorname{div} \mathbf{F}\, dV = \int_0^{2\pi}\int_0^1\int_{r^2}^1 8zr\, dz\, dr\, d\theta$

$= 2\pi \int_0^1 \left(4r - 4r^5\right) dr = \frac{8}{3}\pi$

On S_1: $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + 3\,\mathbf{k}$, $\mathbf{n} = \mathbf{k}$ and

$\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} = \iint_{S_1} 3\, dS = 3\pi$.

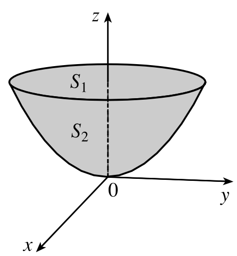
On S_2: $\mathbf{F} = \left(x^3 + xy^2\right)\mathbf{i} + \left(y^3 + yx^2\right)\mathbf{j} + 3\left(x^2+y^2\right)^2\mathbf{k}$,

$-\left(\mathbf{r}_x \times \mathbf{r}_y\right) = 2x\,\mathbf{i} + 2y\,\mathbf{j} - \mathbf{k}$ and

$\iint_{S_2} \mathbf{F}\cdot d\mathbf{S} = \iint_{x^2+y^2\le 1} \left(-x^4 - y^4 - 2x^2y^2\right) dA$

$= -\int_0^{2\pi}\int_0^1 r^5\, dr\, d\theta = -\frac{\pi}{3}$

Hence $\iint_S \mathbf{F}\cdot d\mathbf{S} = 3\pi - \frac{\pi}{3} = \frac{8}{3}\pi$.



2. $\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}\left(3y^2z^3\right) + \frac{\partial}{\partial y}\left(9x^2yz^2\right) + \frac{\partial}{\partial z}\left(4xy^2\right)$

$= 9x^2z^2$

so by the Divergence Theorem,

$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E 9x^2z^2\, dV$

$= \int_{-1}^1\int_{-1}^1\int_{-1}^1 9x^2z^2\, dx\, dy\, dz = 8$

3. $\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}\left(x^2y\right) + \frac{\partial}{\partial y}\left(-x^2z\right) + \frac{\partial}{\partial z}\left(z^2y\right)$

$= 2xy + 2zy$

so by the Divergence Theorem,

$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E (2xy + 2yz)\, dV$

$= \int_0^1\int_0^2\int_0^3 (2xy + 2yz)\, dx\, dy\, dz = 24$

4. $\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(-xz) + \frac{\partial}{\partial y}(-yz) + \frac{\partial}{\partial z}\left(z^2\right)$

$= -z - z + 2z = 0$

so $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\, dV = \iiint_E 0\, dV = 0$.

5. $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E (5y)\, dV$

$= \int_0^1\int_0^{1-x}\int_0^{1-x-y} 5y\, dz\, dy\, dx$

$= \int_0^1\int_0^{1-x}\left[5(1-x)y - 5y^2\right] dy\, dx$

$= \int_0^1\left[\frac{5}{2}(1-x)^3 - \frac{5}{3}(1-x)^3\right] dx = \frac{5}{24}$

6. $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E 3\left(x^2+y^2+z^2\right) dV$

$= \int_0^{2\pi}\int_0^{\pi}\int_0^1 3\rho^4 \sin\phi\, d\rho\, d\phi\, d\theta$

$= 2\pi\int_0^{\pi} \frac{3}{5}\sin\phi\, d\phi = \frac{12}{5}\pi$

7. $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E 3\left(x^2+y^2\right) dV$

$= \int_0^{2\pi}\int_0^2\int_0^{4-r^2} 3r^3\, dz\, dr\, d\theta$

$= 2\pi\int_0^2 \left(12r^3 - 3r^5\right) dr = 32\pi$

8. $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E 2y\, dV$

$= \iint_{x^2+y^2\le 9}\int_{y-3}^0 2y\, dz\, dA$

$= \int_0^{2\pi}\int_0^3\int_{-3+r\sin\theta}^0 \left(2r^2\sin\theta\right) dz\, dr\, d\theta$

$= \int_0^{2\pi}\int_0^3 \left(6r^2\sin\theta - 2r^3\sin^2\theta\right) dr\, d\theta$

$= \int_0^{2\pi}\left[54\sin\theta - \frac{81}{2}\sin^2\theta\right] d\theta = -\frac{81}{2}\pi$

9. $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E x\, dV = \int_0^1\int_0^{2-2x}\int_0^{2-2x-y} x\, dz\, dy\, dx$

$= \int_0^1\int_0^{2-2x}\left[x(2-2x) - xy\right] dy\, dx$

$= \int_0^1\left[x(2-2x)^2 - \frac{1}{2}x(2-2x)^2\right] dx = \frac{1}{6}$

10. $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E (1-1)\, dV = 0$

11. $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \left(x^2+y^2+z\right) dV$

$= \int_0^{2\pi}\int_1^2\int_1^3 \left(r^2+z\right) r\, dz\, dr\, d\theta$

$= 2\pi\int_1^2 \left(2r^3 + 4r\right) dr = 27\pi$

12. $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E 4x^2\, dV$

$= \int_0^{2\pi}\int_0^{\pi}\int_2^3 \left(4\rho^2\sin^2\phi\cos^2\theta\right)\left(\rho^2\sin\phi\right) d\rho\, d\phi\, d\theta$

$= \left[\int_0^{2\pi} 4\cos^2\theta\, d\theta\right]\left[\int_0^{\pi}\sin^3\phi\, d\phi\right]\left[\int_2^3 \rho^4\, d\rho\right]$

$= 4\pi\left(\frac{4}{3}\right)\left[\frac{1}{5}\left(3^5 - 2^5\right)\right] = \frac{3376}{15}\pi$