13.7 Stokes' Theorem

A Click here for answers.

S Click here for solutions.

1–5 Use Stokes' Theorem to evaluate $\iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S}$.

1. $\mathbf{F}(x, y, z) = xyz\,\mathbf{i} + x\,\mathbf{j} + e^{xy}\cos z\,\mathbf{k}$,
S is the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geqslant 0$, oriented upward

2. $\mathbf{F}(x, y, z) = y^2z\,\mathbf{i} + xz\,\mathbf{j} + x^2y^2\,\mathbf{k}$,
S is the part of the paraboloid $z = x^2 + y^2$ that lies inside the cylinder $x^2 + y^2 = 1$, oriented upward

3. $\mathbf{F}(x, y, z) = yz^3\,\mathbf{i} + \sin(xyz)\,\mathbf{j} + x^3\,\mathbf{k}$,
S is the part of the paraboloid $y = 1 - x^2 - z^2$ that lies to the right of the xz-plane, oriented toward the xz-plane

4. $\mathbf{F}(x, y, z) = (x + \tan^{-1}yz)\,\mathbf{i} + y^2z\,\mathbf{j} + z\,\mathbf{k}$,
S is the part of the hemisphere $x = \sqrt{9 - y^2 - z^2}$ that lies inside the cylinder $y^2 + z^2 = 4$, oriented in the direction of the positive x-axis

5. $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + e^z\,\mathbf{j} + xy^2\,\mathbf{k}$,
S consists of the four sides of the pyramid with vertices (0, 0, 0), (1, 0, 0), (0, 0, 1), (1, 0, 1), and (0, 1, 0) that lie to the right of the xz-plane, oriented in the direction of the positive y-axis [*Hint:* Use Equation 3.]

6–8 Use Stokes' Theorem to evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$. In each case C is oriented counterclockwise as viewed from above.

6. $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + 2xy\,\mathbf{j} + 3xy\,\mathbf{k}$,
C is the boundary of the part of the plane $3x + y + z = 3$ in the first octant

7. $\mathbf{F}(x, y, z) = z^2\,\mathbf{i} + y^2\,\mathbf{j} + xy\,\mathbf{k}$,
C is the triangle with vertices (1, 0, 0), (0, 1, 0), and (0, 0, 2)

8. $\mathbf{F}(x, y, z) = 2z\,\mathbf{i} + 4x\,\mathbf{j} + 5y\,\mathbf{k}$,
C is the curve of intersection of the plane $z = x + 4$ and the cylinder $x^2 + y^2 = 4$

9–12 Verify that Stokes' Theorem is true for the given vector field $\mathbf{F}$ and surface S.

9. $\mathbf{F}(x, y, z) = 3y\,\mathbf{i} + 4z\,\mathbf{j} - 6x\,\mathbf{k}$,
S is the part of the paraboloid $z = 9 - x^2 - y^2$ that lies above the xy-plane, oriented upward

10. $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + xz\,\mathbf{k}$,
S is the hemisphere $z = \sqrt{a^2 - x^2 - y^2}$, oriented upward

11. $\mathbf{F}(x, y, z) = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$,
S is the part of the plane $x + y + z = 1$ that lies in the first octant, oriented upward

12. $\mathbf{F}(x, y, z) = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$,
S is the helicoid with vector equation
$\mathbf{r}(u, v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + v\,\mathbf{k}$, $0 \leqslant u \leqslant 1$, $0 \leqslant v \leqslant \pi$

Answers

E Click here for exercises.

S Click here for solutions.

1. π

2. π

3. $\frac{3\pi}{4}$

4. -4π

5. 0

6. $\frac{7}{2}$

7. $\frac{4}{3}$

8. -4π

Solutions

E Click here for exercises.

A Click here for answers.

1. The boundary curve is C: $x^2+y^2=1$, $z=0$ oriented in the counterclockwise direction. The vector equation of C is $\mathbf{r}(t)=\cos t\,\mathbf{i}+\sin t\,\mathbf{j}$, $0\le t\le 2\pi$. Then $\mathbf{F}(\mathbf{r}(t))=\cos t\,\mathbf{j}+e^{\cos t\sin t}\,\mathbf{k}$ and $\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)=\cos^2 t$. Hence $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}=\oint_C\mathbf{F}\cdot d\mathbf{r}=\int_0^{2\pi}\cos^2 t\,dt=\int_0^{2\pi}\frac{1}{2}(1+\cos 2t)\,dt=\pi$.

2. The paraboloid intersects the cylinder in the circle $x^2+y^2=1$, $z=1$ and the vector equation is $\mathbf{r}(t)=\cos t\,\mathbf{i}+\sin t\,\mathbf{j}+\mathbf{k}$, $0\le t\le 2\pi$. Then $\mathbf{F}(\mathbf{r}(t))=\sin^2 t\,\mathbf{i}+\cos t\,\mathbf{j}+\cos^2 t\sin^2 t\,\mathbf{k}$ and $\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)=-\sin^3 t+\cos^2 t$. Hence $\iint_S\operatorname{curl}\mathbf{F}\cdot d\mathbf{S}=\oint_C\mathbf{F}\cdot d\mathbf{r}=\int_0^{2\pi}\left(\cos^2 t-\sin^3 t\right)\,dt=\pi$.

3. C is the circle $x^2+z^2=1$, $y=0$ and the vector equation is $\mathbf{r}(t)=\cos t\,\mathbf{i}+\sin t\,\mathbf{k}$, $0\le t\le 2\pi$ since the surface is oriented toward the xy-plane. Then $\mathbf{F}(\mathbf{r}(t))=\cos^3 t\,\mathbf{k}$ and $\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)=\cos^4 t$. Hence

$$\iint_S\operatorname{curl}\mathbf{F}\cdot d\mathbf{S}=\oint_C\mathbf{F}\cdot d\mathbf{r}=\int_0^{2\pi}\cos^4 t\,dt=\int_0^{2\pi}\left[\tfrac{3}{8}+\tfrac{1}{2}\cos 2t+\tfrac{1}{8}\cos 4t\right]dt=\tfrac{3\pi}{4}$$

4. C is the circle $y^2+z^2=4$, $x=\sqrt{5}$ with vector equation $\mathbf{r}(t)=\sqrt{5}\,\mathbf{i}+2\cos t\,\mathbf{j}+2\sin t\,\mathbf{k}$, $0\le t\le 2\pi$. Then

$$\mathbf{F}(\mathbf{r}(t))=\left[\sqrt{5}+\tan^{-1}(4\cos t\sin t)\right]\mathbf{i}+8\cos^2 t\sin t\,\mathbf{j}+2\sin t\,\mathbf{k}$$

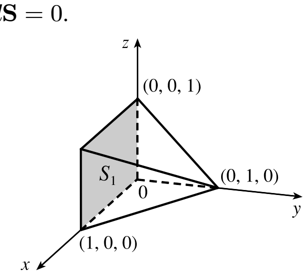
and

$$\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)=-16\cos^2 t\sin^2 t+4\sin t\cos t=-2+2\cos 2t+2\sin 2t$$

Thus

$$\iint_S\operatorname{curl}\mathbf{F}\cdot d\mathbf{S}=\oint_C\mathbf{F}\cdot d\mathbf{r}=2\int_0^{2\pi}(-1+\cos 2t+\sin 2t)\,dt=-4\pi$$

5. Here S consists of the 4 sides of the pyramid but not the base in the xz-plane. Call the base S_1. Then $\iint_S\operatorname{curl}\mathbf{F}\cdot d\mathbf{S}=\oint_C\mathbf{F}\cdot d\mathbf{r}$ where C is the boundary of the base. To avoid calculating four line integrals, apply Stokes' Theorem again. Then $\oint_C\mathbf{F}\cdot d\mathbf{r}=\iint_{S_1}\operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$. But $\operatorname{curl}\mathbf{F}=(2xy-e^z)\,\mathbf{i}-y^2\,\mathbf{j}-x\,\mathbf{k}$ and $\mathbf{n}=\mathbf{j}$, so $\operatorname{curl}\mathbf{F}\cdot\mathbf{n}=-y^2=0$ on S_1, $\iint_{S_1}\operatorname{curl}\mathbf{F}\cdot d\mathbf{S}=0$ and $\iint_S\operatorname{curl}\mathbf{F}\cdot d\mathbf{S}=0$.



6. $\operatorname{curl}\mathbf{F}=3x\,\mathbf{i}+(x-3y)\,\mathbf{j}+2y\,\mathbf{k}$, $\mathbf{n}=\dfrac{1}{\sqrt{11}}(3\,\mathbf{i}+\mathbf{j}+\mathbf{k})$

and

$$\begin{aligned}\oint_C\mathbf{F}\cdot d\mathbf{r}&=\iint_S\operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS\\&=\int_0^1\int_0^{3-3x}\tfrac{1}{\sqrt{11}}\left[9x+(x-3y)+2y\right]\left(\sqrt{11}\right)dy\,dx\\&=\int_0^1\int_0^{3-3x}(10x-y)\,dy\,dx\\&=\int_0^1\left[10\left(3x-3x^2\right)-\tfrac{1}{2}(3-3x)^2\right]dx\\&=\left[15x^2-10x^3+\tfrac{3}{2}\left(1-x^3\right)\right]_0^1=\tfrac{7}{2}\end{aligned}$$

7. The triangle is in the plane $2x+2y+z=2$ with normal $\mathbf{n}=\frac{1}{3}(2\,\mathbf{i}+2\,\mathbf{j}+\mathbf{k})$, $\operatorname{curl}\mathbf{F}=x\,\mathbf{i}+(2z-y)\,\mathbf{j}$, $\operatorname{curl}\mathbf{F}\cdot\mathbf{n}=\frac{1}{3}(2x+4z-2y)=\frac{1}{3}(8-6x-10y)$ and $dS=3\,dx\,dy$. So

$$\begin{aligned}\oint_C\mathbf{F}\cdot d\mathbf{r}&=\iint_S\operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS\\&=\int_0^1\int_0^{1-x}(8-6x-10y)\,dy\,dx\\&=\int_0^1\left[8(1-x)-6\left(x-x^2\right)-5(1-x)^2\right]dx\\&=\left[8x-7x^2+2x^3+\tfrac{5}{3}(1-x)^3\right]_0^1=\tfrac{4}{3}\end{aligned}$$

8. The curve of intersection is an ellipse in the plane $z=x+4$ with unit normal $\mathbf{n}=\frac{1}{\sqrt{2}}(-\mathbf{i}+\mathbf{k})$ and $\operatorname{curl}\mathbf{F}=5\,\mathbf{i}+2\,\mathbf{j}+4\,\mathbf{k}$ so $\operatorname{curl}\mathbf{F}\cdot\mathbf{n}=-\frac{1}{\sqrt{2}}$. Then

$$\begin{aligned}\oint_C\mathbf{F}\cdot d\mathbf{r}&=-\iint_S\tfrac{1}{\sqrt{2}}\,dS\\&=-\tfrac{1}{\sqrt{2}}\cdot(\text{surface area of planar ellipse})\\&=-\tfrac{1}{\sqrt{2}}\pi(2)\left(2\sqrt{2}\right)=-4\pi\end{aligned}$$

Recall that the area of an ellipse with semiaxes a and b is πab.

9. The boundary curve C is the circle $x^2+y^2=9$, $z=0$ oriented in the counterclockwise direction as viewed from $(0,0,1)$. Then $\mathbf{r}(t)=3\cos t\,\mathbf{i}+3\sin t\,\mathbf{j}$, $0\le t\le 2\pi$, so $\mathbf{F}(\mathbf{r}(t))=9\sin t\,\mathbf{i}-18\cos t\,\mathbf{k}$ and $\mathbf{F}\cdot\mathbf{r}'(t)=-27\sin^2 t$. Thus $\oint_C\mathbf{F}\cdot d\mathbf{r}=\int_0^{2\pi}\left(-27\sin^2 t\right)dt=-27\pi$. Now $\operatorname{curl}\mathbf{F}=-4\,\mathbf{i}+6\,\mathbf{j}-3\,\mathbf{k}$, $\mathbf{r}_x\times\mathbf{r}_y=2x\,\mathbf{i}+2y\,\mathbf{j}+\mathbf{k}$, so

$$\begin{aligned}\iint_S\operatorname{curl}\mathbf{F}\cdot d\mathbf{S}&=\iint_{x^2+y^2\le 9}(-8x+12y-3)\,dA\\&=\int_0^{2\pi}\int_0^3(-8r\cos\theta+12r\sin\theta-3)\,r\,dr\,d\theta\\&=\int_0^3(-3r)(2\pi)\,dr=-27\pi\end{aligned}$$

10. C: $x^2+y^2=a^2$, $z=0$,

$\oint_C \mathbf{F}\cdot d\mathbf{r}=\int_0^{2\pi}\left(a^2\sin t\cos t\right)(-a\sin t)\,dt$

$=-\frac{1}{3}a^3\sin^3 t\Big]_0^{2\pi}=0$

Then $\operatorname{curl}\mathbf{F}=-y\,\mathbf{i}-z\,\mathbf{j}-x\,\mathbf{k}$,

$$\mathbf{r}_x\times\mathbf{r}_y=\frac{x}{(a^2-x^2-y^2)^{1/2}}\,\mathbf{i}+\frac{y}{(a^2-x^2-y^2)^{1/2}}\,\mathbf{j}+\mathbf{k}.$$

Hence $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$

$$=\iint_{x^2+y^2\le a^2}\left[-\frac{yx}{(a^2-x^2-y^2)^{1/2}}-y-x\right]dA$$

$$=-\int_0^a\int_0^{2\pi}\left[\frac{r^2\cos\theta\sin\theta}{\sqrt{a^2-r^2}}+r\sin\theta+r\cos\theta\right]r\,d\theta\,dr=0$$

since $\int_0^{2\pi}\sin\theta\,d\theta=\int_0^{2\pi}\cos\theta\,d\theta=\int_0^{2\pi}\cos\theta\sin\theta\,d\theta=0$.

Notice that for this reason, it's much easier to integrate with respect to θ first.

11. The x-, y-, and z-intercepts of the plane are all 1, so C consists of the three line segments

C_1: $\mathbf{r}_1(t)=(1-t)\,\mathbf{i}+t\,\mathbf{j}$, $0\le t\le 1$,

C_2: $\mathbf{r}_2(t)=(1-t)\,\mathbf{j}+t\,\mathbf{k}$, $0\le t\le 1$, and

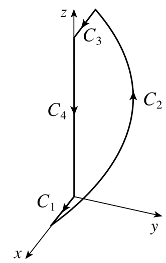
C_3: $\mathbf{r}_3(t)=t\,\mathbf{i}+(1-t)\,\mathbf{k}$, $0\le t\le 1$. Then

$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\int_0^1[t\,\mathbf{i}+(1-t)\,\mathbf{k}]\cdot(-\mathbf{i}+\mathbf{j})\,dt+\int_0^1[(1-t)\,\mathbf{i}+t\,\mathbf{j}]\cdot(-\mathbf{j}+\mathbf{k})\,dt+\int_0^1[(1-t)\,\mathbf{j}+t\,\mathbf{k}]\cdot(\mathbf{i}-\mathbf{k})\,dt$$

$$=\int_0^1(-3t)\,dt=-\tfrac{3}{2}$$

Now $\operatorname{curl}\mathbf{F}=-\mathbf{i}-\mathbf{j}-\mathbf{k}$ and $\mathbf{r}_x\times\mathbf{r}_y=\mathbf{i}+\mathbf{j}+\mathbf{k}$. Hence $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}=\int_0^1\int_0^{1-x}(-3)\,dy\,dx=-\frac{3}{2}$.

12. The equations of the helicoid are $\mathbf{r}(u,v)=u\cos v\,\mathbf{i}+u\sin v\,\mathbf{j}+v\,\mathbf{k}$, $0\le u\le 1$, $0\le v\le\pi$. The boundary curve C of the helicoid consists of the four curves C_1: $v=0$, $0\le u\le 1$, C_2: $u=1$, $0\le v\le\pi$, C_3: $v=\pi$, $u=1$ to $u=0$, C_4: $u=0$, $v=\pi$ to $v=0$. Then



$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\int_0^1(u\,\mathbf{k})\cdot(\mathbf{i})\,du+\int_0^\pi(\sin v\,\mathbf{i}+v\,\mathbf{j}+\cos v\,\mathbf{k})\cdot(-\sin v\,\mathbf{i}+\cos v\,\mathbf{j}+\mathbf{k})\,dv+\int_1^0(-u\,\mathbf{k})\cdot(-\mathbf{i})\,du+\int_\pi^0(v\,\mathbf{j})\cdot(\mathbf{k})\,dv$$

$$=\int_0^\pi\left(-\sin^2 v+v\cos v+\cos v\right)dv$$

$$=\left[-\tfrac{1}{2}(v-\sin v\cos v)+v\sin v+\cos v+\sin v\right]_0^\pi$$

$$=-\tfrac{\pi}{2}-2=-\tfrac{1}{2}(\pi+4)$$

Now $\operatorname{curl}\mathbf{F}=-\mathbf{i}-\mathbf{j}-\mathbf{k}$ and $\mathbf{r}_u\times\mathbf{r}_v=\sin v\,\mathbf{i}-\cos v\,\mathbf{j}+u\,\mathbf{k}$. Hence

$$\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}=\int_0^\pi\int_0^1(-\sin v+\cos v-u)\,du\,dv=\int_0^\pi\left[-\sin v+\cos v-\tfrac{1}{2}\right]dv=-2-\tfrac{\pi}{2}=-\tfrac{1}{2}(\pi+4)$$