13.6 Surface Integrals

A Click here for answers.

S Click here for solutions.

1–6 Evaluate the surface integral.

1. $\iint_S y\, dS$,
S is the part of the plane $3x + 2y + z = 6$ that lies in the first octant

2. $\iint_S xz\, dS$,
S is the triangular with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$

3. $\iint_S x\, dS$,
S is the surface $y = x^2 + 4z$, $0 \leqslant x \leqslant 2$, $0 \leqslant z \leqslant 2$

4. $\iint_S (y^2 + z^2)\, dS$,
S is the part of the paraboloid $x = 4 - y^2 - z^2$ that lies in front of the plane $x = 0$

5. $\iint_S yz\, dS$,
S is the part of the plane $z = y + 3$ that lies inside the cylinder $x^2 + y^2 = 1$

6. $\iint_S yz\, dS$,
S is the surface with parametric equations $x = uv$, $y = u + v$, $z = u - v$, $u^2 + v^2 \leqslant 1$

7–12 Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ for the given vector field $\mathbf{F}$ and the oriented surface S. In other words, find the flux of $\mathbf{F}$ across S. For closed surfaces, use the positive (outward) orientation.

7. $\mathbf{F}(x, y, z) = e^y\,\mathbf{i} + ye^x\,\mathbf{j} + x^2y\,\mathbf{k}$,
S is the part of the paraboloid $z = x^2 + y^2$ that lies above the square $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$ and has upward orientation

8. $\mathbf{F}(x, y, z) = x^2y\,\mathbf{i} - 3xy^2\,\mathbf{j} + 4y^3\,\mathbf{k}$,
S is the part of the elliptic paraboloid $z = x^2 + y^2 - 9$ that lies below the square $0 \leqslant x \leqslant 2$, $0 \leqslant y \leqslant 1$ and has downward orientation

9. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + xy\,\mathbf{j} + xz\,\mathbf{k}$,
S is the surface of Problem 1 with upward orientation

10. $\mathbf{F}(x, y, z) = -x\,\mathbf{i} - y\,\mathbf{j} + z^2\,\mathbf{k}$,
S is the part of the cone $z = \sqrt{x^2 + y^2}$ between the planes $z = 1$ and $z = 2$ with upward orientation

11. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$,
S is the sphere $x^2 + y^2 + z^2 = 9$

12. $\mathbf{F}(x, y, z) = -y\,\mathbf{i} + x\,\mathbf{j} + 3z\,\mathbf{k}$,
S is the hemisphere $z = \sqrt{16 - x^2 - y^2}$ with upward orientation

Answers

E Click here for exercises.

S Click here for solutions.

1. $3\sqrt{14}$

2. $\frac{\sqrt{3}}{24}$

3. $\frac{33\sqrt{33}-17\sqrt{17}}{6}$

4. $\frac{\pi}{60}\left(391\sqrt{17}+1\right)$

5. $\frac{\sqrt{2}\pi}{4}$

6. 0

7. $\frac{1}{6}\left(11-10e\right)$

8. -1

9. 12

10. $\frac{73}{6}\pi$

11. 108π

12. 128π

Solutions

E Click here for exercises.

A Click here for answers.

1. $\mathbf{r}(x,y) = x\,\mathbf{i} + y\,\mathbf{j} + (6-3x-2y)\,\mathbf{k}$, $\mathbf{r}_x \times \mathbf{r}_y = 3\,\mathbf{i} + 2\,\mathbf{j} + \mathbf{k}$ (the normal to the plane) and $|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{14}$. The given plane meets the first octant in the line $3x+2y=6$, $z=0$, $x \ge 0$, $y \ge 0$, so $D = \left\{(x,y) \mid 0 \le x \le \frac{1}{3}(6-2y), 0 \le y \le 3\right\}$. Then

$$\iint_S y\,dS = \int_0^3 \int_0^{(6-2y)/3} y\sqrt{14}\;dx\,dy = \sqrt{14}\int_0^3 \left(2y - \tfrac{2}{3}y^2\right)\,dy = 3\sqrt{14}$$

2. $\mathbf{r}(x,y) = x\,\mathbf{i} + y\,\mathbf{j} + (1-x-y)\,\mathbf{k}$, $0 \le x \le 1-y$, $0 \le y \le 1$, $|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{3}$. Then

$$\iint_S xz\,dS = \int_0^1 \int_0^{1-y} \left(x(1-y) - x^2\right)\sqrt{3}\;dx\,dy = \sqrt{3}\int_0^1 \left[\tfrac{1}{2}(1-y)^3 - \tfrac{1}{3}(1-y)^3\right]\,dy = \tfrac{\sqrt{3}}{24}$$

3. Using x and z as parameters, we have $\mathbf{r}(x,z) = x\,\mathbf{i} + \left(x^2+4z\right)\mathbf{j} + z\,\mathbf{k}$, $0 \le x \le 2$, $0 \le z \le 2$. Then $\mathbf{r}_x \times \mathbf{r}_z = (\mathbf{i} + 2x\,\mathbf{j}) \times (4\,\mathbf{j} + \mathbf{k}) = 2x\,\mathbf{i} - \mathbf{j} + 4\,\mathbf{k}$ and $|\mathbf{r}_x \times \mathbf{r}_z| = \sqrt{4x^2+17}$. Thus

$$\iint_S x\,dS = \int_0^2\int_0^2 x\sqrt{4x^2+17}\;dx\,dz = \int_0^2 dz\;\int_0^2 x\sqrt{4x^2+17}\;dx = 2\left[\tfrac{1}{12}\left(4x^2+17\right)^{3/2}\right]_0^2 = \tfrac{33\sqrt{33}-17\sqrt{17}}{6}$$

4. $\mathbf{r}(y,z) = \left(4-y^2-z^2\right)\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, $0 \le y^2+z^2 \le 4$, so $\mathbf{r}_y \times \mathbf{r}_z = (-2y\,\mathbf{i} + \mathbf{j}) \times (-2z\,\mathbf{i} + \mathbf{k}) = \mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}$ and $|\mathbf{r}_y \times \mathbf{r}_z| = \sqrt{4y^2+4z^2+1}$. Then

$$\begin{aligned}\iint_S \left(y^2+z^2\right)dS &= \iint_{y^2+z^2\le 4}\left(y^2+z^2\right)\sqrt{4y^2+4z^2+1}\,dA \\ &= \int_0^{2\pi}\int_0^2 r^2\sqrt{4r^2+1}\,r\,dr\,d\theta \\ &= \int_0^{2\pi} d\theta\;\int_0^2 r^3\sqrt{4r^2+1}\,dr\end{aligned}$$

Substituting $u = 4r^2+1$, so $du = 8r\,dr$ and $r = \frac{1}{4}(u-1)$, gives

$$\begin{aligned}\iint_S \left(y^2+z^2\right)dS &= 2\pi\int_1^{17}\tfrac{1}{8}\tfrac{1}{4}(u-1)\sqrt{u}\,du \\ &= \tfrac{\pi}{16}\left[\tfrac{2}{5}u^{5/2} - \tfrac{2}{3}u^{3/2}\right]_1^{17} \\ &= \tfrac{\pi}{16}\left[\tfrac{2}{5}\left(289\sqrt{17}-1\right) - \tfrac{2}{3}\left(17\sqrt{17}-1\right)\right] \\ &= \tfrac{\pi}{16}\left(\tfrac{1564}{15}\sqrt{17} + \tfrac{4}{15}\right) = \tfrac{\pi}{60}\left(391\sqrt{17}+1\right)\end{aligned}$$

5. S is the part of the plane $z = y+3$ over the disk $D = \left\{(x,y) \mid x^2+y^2 \le 1\right\}$. Thus

$$\begin{aligned}\iint_S yz\,dS &= \iint_D y(y+3)\sqrt{(0)^2+(1)^2+1}\,dA \\ &= \sqrt{2}\int_0^{2\pi}\int_0^1 r\sin\theta\,(r\sin\theta+3)\,r\,dr\,d\theta \\ &= \sqrt{2}\int_0^{2\pi}\left[\tfrac{1}{4}r^4\sin^2\theta + r^3\sin\theta\right]_{r=0}^{r=1}\,d\theta \\ &= \sqrt{2}\int_0^{2\pi}\left(\tfrac{1}{4}\sin^2\theta + \sin\theta\right)d\theta \\ &= \sqrt{2}\left[\tfrac{1}{4}\left(\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right) - \cos\theta\right]_0^{2\pi} = \tfrac{\sqrt{2}\pi}{4}\end{aligned}$$

6. $\mathbf{r}(u,v) = uv\,\mathbf{i} + (u+v)\,\mathbf{j} + (u-v)\,\mathbf{k}$, $u^2+v^2 \le 1$ and $|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{4+2u^2+2v^2}$ (see Exercise 16.6.45 in the text). Then

$$\begin{aligned}\iint_S yx\,dS &= \iint_{u^2+v^2\le 1}\left(u^2-v^2\right)\sqrt{4+2u^2+2v^2}\,dA \\ &= \int_0^{2\pi}\int_0^1 r^2\left(\cos^2\theta - \sin^2\theta\right)\sqrt{4+2r^2}\,r\,dr\,d\theta \\ &= \left[\int_0^{2\pi}\left(\cos^2\theta-\sin^2\theta\right)d\theta\right]\left[\int_0^1 r^3\sqrt{4+2r^2}\,dr\right]\end{aligned}$$

$= 0$ since the first integral is 0.

7. $\mathbf{F}(\mathbf{r}(x,y)) = e^y\,\mathbf{i} + ye^x\,\mathbf{j} + x^2y\mathrm{k}$ and $\mathbf{r}_x \times \mathbf{r}_y = -2x\,\mathbf{i} - 2y\,\mathbf{j} + \mathbf{k}$. Then $\mathbf{F}(\mathbf{r}(x,y)) \cdot (\mathbf{r}_x \times \mathbf{r}_y) = -2xe^y - 2y^2e^x + x^2y$ and

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \int_0^1\int_0^1\left(-2xe^y - 2y^2e^x + x^2y\right)dx\,dy \\ &= \int_0^1\left(-e^y - 2ey^2 + \tfrac{1}{3}y + 2y^2\right)dy \\ &= \tfrac{1}{6}(11-10e)\end{aligned}$$

8. $\mathbf{F}(\mathbf{r}(x,y)) = x^2y\,\mathbf{i} - 3xy^2\,\mathbf{j} + 4y^3\,\mathbf{k}$, $\mathbf{r}_y \times \mathbf{r}_x = 2x\,\mathbf{i} + 2y\,\mathbf{j} - \mathbf{k}$ (since downward), and $\mathbf{F}(\mathbf{r}(x,y)) \cdot (\mathbf{r}_x \times \mathbf{r}_y) = 2x^3y - 6xy^3 - 4y^3$. Hence

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \int_0^2\int_0^1\left(2x^3y - 6xy^3 - 4y^3\right)dy\,dx \\ &= \int_0^2\left(x^3 - \tfrac{3}{2}x - 1\right)dx = -1\end{aligned}$$

9. As in Problem 1, $D = \left\{(x,y) \mid 0 \le x \le 2, 0 \le y \le \frac{1}{2}(6-3x)\right\}$.

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \int_0^2\int_0^{(6-3x)/2}\left[x\,\mathbf{i} + xy\,\mathbf{j} + x(6-3x-2y)\,\mathbf{k}\right]\cdot(3\,\mathbf{i} + 2\,\mathbf{j} + \mathbf{k})\,dy\,dx \\ &= \int_0^2\int_0^{(6-3x)/2}\left(9x - 3x^2\right)dy\,dx \\ &= \int_0^2\left[27x - \tfrac{45}{2}x^2 + \tfrac{9}{2}x^3\right]dx = 12\end{aligned}$$

10. $\mathbf{r}_x \times \mathbf{r}_y = -\dfrac{x}{\sqrt{x^2+y^2}}\,\mathbf{i} - \dfrac{y}{\sqrt{x^2+y^2}}\,\mathbf{j} + \mathbf{k}$ (since upward) and

$\mathbf{F}(\mathbf{r}(x,y)) \cdot (\mathbf{r}_x \times \mathbf{r}_y) = \dfrac{x^2+y^2}{\sqrt{x^2+y^2}} + x^2 + y^2$ where $1 \le x^2+y^2 \le 4$. Then

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_{1\le x^2+y^2\le 4}\left(\sqrt{x^2+y^2} + x^2 + y^2\right)dA \\ &= \int_0^{2\pi}\int_1^2\left(r+r^2\right)r\,dr\,d\theta = 2\pi\left(\tfrac{73}{12}\right) = \tfrac{73}{6}\pi\end{aligned}$$

11. $\mathbf{F}(\mathbf{r}(\phi,\theta)) = 3\sin\phi\cos\theta\,\mathbf{i} + 3\sin\phi\sin\theta\,\mathbf{j} + 3\cos\phi\,\mathbf{k}$ and $\mathbf{r}_\phi \times \mathbf{r}_\theta = 9\sin^2\phi\cos\theta\,\mathbf{i} + 9\sin^2\phi\sin\theta\,\mathbf{j} + 9\sin\phi\cos\phi\,\mathbf{k}$. Then

$$\begin{aligned}\mathbf{F}(\mathbf{r}(\phi,\theta)) \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) &= 27\sin^3\phi\cos^2\theta + 27\sin^3\phi\sin^2\theta + 27\sin\phi\cos^2\phi \\ &= 27\sin\phi\end{aligned}$$

and

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{\pi} 27\sin\phi\,d\phi\,d\theta = (2\pi)(54) = 108\pi.$$

12. $\mathbf{F}(\mathbf{r}(\phi,\theta)) = -4\sin\phi\sin\theta\,\mathbf{i} + 4\sin\phi\cos\theta\,\mathbf{j} + 12\cos\phi\,\mathbf{k}$

and

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = 16\sin^2\phi\cos\theta\,\mathbf{i} + 16\sin^2\phi\sin\theta\,\mathbf{j} + 16\sin\phi\cos\phi\,\mathbf{k}$$

Then

$$\begin{aligned}\mathbf{F}(\mathbf{r}(\phi,\theta))\cdot(\mathbf{r}_\phi \times \mathbf{r}_\theta) &= -64\sin^3\phi\sin\theta\cos\theta + 64\sin^3\phi\sin\theta\cos\theta + 192\sin\phi\cos^2\phi \\ &= 192\sin\phi\cos^2\phi\end{aligned}$$

and

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \int_0^{2\pi}\int_0^{\pi/2} 192\sin\phi\cos^2\phi\,d\phi\,d\theta \\ &= 2\pi\left[-64\cos^3\phi\right]_0^{\pi/2} = 128\pi\end{aligned}$$