13.4 Green's Theorem

A Click here for answers.

S Click here for solutions.

1–4 Evaluate the line integral by two methods: (a) directly and (b) using Green's Theorem.

1. $\oint_C x^2y\,dx + xy^3\,dy$,
C is the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$

2. $\oint_C x\,dx - x^2y^2\,dy$,
C is the triangle with vertices $(0, 0)$, $(1, 1)$, and $(0, 1)$

3. $\oint_C (x + 2y)\,dx + (x - 2y)\,dy$,
C consists of the arc of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$ followed by the line segment from $(1, 1)$ to $(0, 0)$

4. $\oint_C (x^2 + y^2)\,dx + 2xy\,dy$,
C consists of the arc of the parabola $y = x^2$ from $(0, 0)$ to $(2, 4)$ and the line segments from $(2, 4)$ to $(0, 4)$ and from $(0, 4)$ to $(0, 0)$

5–16 Use Green's Theorem to evaluate the line integral along the given positively oriented curve.

5. $\int_C xy\,dx + y^5\,dy$,
C is the triangle with vertices $(0, 0)$, $(2, 0)$, and $(2, 1)$

6. $\int_C x^2y\,dx + xy^5\,dy$,
C is the square with vertices $(\pm1, \pm1)$

7. $\int_C x^2\,dx + y^2\,dy$, C is the curve $x^6 + y^6 = 1$

8. $\int_C x^2y\,dx - 3y^2\,dy$, C is the circle $x^2 + y^2 = 1$

9. $\int_C 2xy\,dx + x^2\,dy$, C is the cardioid $r = 1 + \cos\theta$

10. $\int_C (xy + e^{x^2})\,dx + (x^2 - \ln(1 + y))\,dy$,
C consists of the line segment from $(0, 0)$ to $(\pi, 0)$ and the curve $y = \sin x$, $0 \leqslant x \leqslant \pi$

11. $\int_C (y^2 - \tan^{-1}x)\,dx + (3x + \sin y)\,dy$,
C is the boundary of the region enclosed by the parabola $y = x^2$ and the line $y = 4$

12. $\int_C xy\,dx + 2x^2\,dy$,
C consists of the line segment from $(-2, 0)$ to $(2, 0)$ and the top half of the circle $x^2 + y^2 = 4$

13. $\int_C (x^3 - y^3)\,dx + (x^3 + y^3)\,dy$,
C is the boundary of the region between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$

14. $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = (y^2 - x^2y)\,\mathbf{i} + xy^2\,\mathbf{j}$ and
C consists of the circle $x^2 + y^2 = 4$ from $(2, 0)$ to $(\sqrt{2}, \sqrt{2})$ and the line segments from $(\sqrt{2}, \sqrt{2})$ to $(0, 0)$ and from $(0, 0)$ to $(2, 0)$

15. $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = y^6\,\mathbf{i} + xy^5\,\mathbf{j}$ and
C is the ellipse $4x^2 + y^2 = 1$

16. $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = x^3y\,\mathbf{i} + x^4\,\mathbf{j}$ and
C is the curve $x^4 + y^4 = 1$

17–18 Find the area of the given region using one of the formulas in Equations 5.

17. The region bounded by the hypocycloid with vector equation $\mathbf{r}(t) = \cos^3 t\,\mathbf{i} + \sin^3 t\,\mathbf{j}$, $0 \leqslant t \leqslant 2\pi$

18. The region bounded by the curve with vector equation $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin^3 t\,\mathbf{j}$, $0 \leqslant t \leqslant 2\pi$

Answers

E Click here for exercises.

S Click here for solutions.

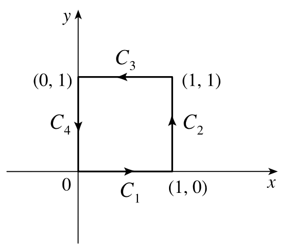
1. $-\frac{1}{12}$
2. $-\frac{1}{5}$
3. $-\frac{1}{6}$
4. 0
5. $-\frac{4}{3}$
6. $-\frac{4}{3}$
7. 0
8. $-\frac{\pi}{4}$
9. 0
10. π
11. $-\frac{96}{5}$
12. 0
13. 120π
14. $\pi + \frac{8}{3}\left(\sqrt{2} - 2\right)$
15. 0
16. 0
17. $\frac{3}{8}\pi$
18. $\frac{3}{4}\pi$

Solutions

E Click here for exercises.

A Click here for answers.

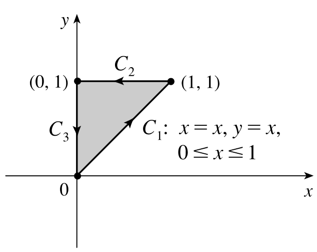
1. (a)



$$\oint_C x^2y\,dx + xy^3\,dy = \oint_{C_1+C_2+C_3+C_4} x^2y\,dx + xy^3\,dy$$
$$= \int_0^1 0\,dx + \int_0^1 y^3\,dy + \int_1^0 x^2\,dx + \int_1^0 0\,dy$$
$$= \tfrac{1}{4} - \tfrac{1}{3} = -\tfrac{1}{12}$$

(b) $\oint_C x^2y\,dx + xy^3\,dy = \int_0^1\int_0^1 \left(y^3 - x^2\right) dx\,dy$
$$= \int_0^1 \left(y^3 - \tfrac{1}{3}\right) dy = \tfrac{1}{4} - \tfrac{1}{3} = -\tfrac{1}{12}$$

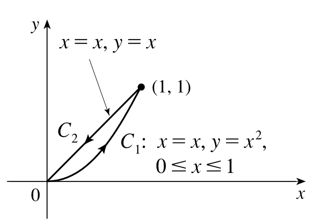
2. (a)



$$\oint_C x\,dx - x^2y^2\,dy = \oint_{C_1+C_2+C_3} x\,dx - x^2y^2\,dy$$
$$= \int_0^1 \left(x - x^4\right) dx + \int_1^0 x\,dx + \int_1^0 0\,dy$$
$$= \tfrac{1}{2} - \tfrac{1}{5} - \tfrac{1}{2} = -\tfrac{1}{5}$$

(b) $\oint_C x\,dx - x^2y^2\,dy = \int_0^1\int_x^1 \left(-2xy^2 - 0\right) dy\,dx$
$$= \int_0^1 \tfrac{2}{3}\left(x^4 - x\right) dx = \tfrac{2}{3}\left(-\tfrac{3}{10}\right) = -\tfrac{1}{5}$$

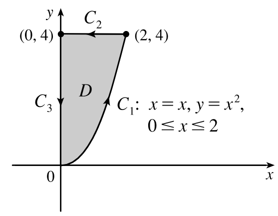
3. (a)



$$\oint_C (x+2y)\,dx + (x-2y)\,dy$$
$$= \oint_{C_1+C_2} (x+2y)\,dx + (x-2y)\,dy$$
$$= \int_0^1 \left[x + 2x^2 + \left(x - 2x^2\right)(2x)\right] dx + \int_1^0 [3x + (-x)]\,dx$$
$$= \int_0^1 \left(x + 4x^2 - 4x^3\right) dx + \int_1^0 2x\,dx$$
$$= \left(\tfrac{1}{2} + \tfrac{4}{3} - 1\right) - 1 = -\tfrac{1}{6}$$

(b) $\oint_C (x+2y)\,dx + (x-2y)\,dy = \int_0^1\int_{x^2}^x (1-2)\,dy\,dx$
$$= \int_0^1 \left(x^2 - x\right) dx = \tfrac{1}{3} - \tfrac{1}{2} = -\tfrac{1}{6}$$

4. (a)



$$\oint_C \left(x^2 + y^2\right) dx + 2xy\,dy$$
$$= \oint_{C_1+C_2+C_3} \left(x^2 + y^2\right) dx + 2xy\,dy$$
$$= \int_0^2 \left[\left(x^2 + x^4\right) + \left(2x^3\right)(2x)\right] dx + \int_2^0 \left(x^2 + 16\right) dx + \int_4^0 0\,dy$$
$$= \tfrac{8}{3} + 32 - \tfrac{8}{3} - 32 = 0$$

(b) $\oint_C \left(x^2 + y^2\right) dx + 2xy\,dy$
$$= \iint_D \left[\frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}\left(x^2 + y^2\right)\right] dA$$
$$= \iint_D (2y - 2y)\,dA = \iint_D (0)\,dA = 0$$

5. $\oint_C xy\,dx + y^5\,dy = \int_0^2\int_0^{x/2} (0 - x)\,dy\,dx$
$$= \int_0^2 \left(-\tfrac{1}{2}x^2\right) dx = -\tfrac{4}{3}$$

6. $\int_{-1}^1\int_{-1}^1 \left(y^5 - x^2\right) dy\,dx = \int_{-1}^1 \left(-\tfrac{2}{3}\right) dx = -\tfrac{4}{3}$

7. $\iint_D (0 - 0)\,dA = 0$

8. $\iint_{x^2+y^2\le 1} \left(0 - x^2\right) dA = -\int_0^{2\pi}\int_0^1 r^3\cos^2\theta\,dr\,d\theta$
$$= -\pi\left(\tfrac{1}{4}\right) = -\tfrac{\pi}{4}$$

9. $\iint_D (2x - 2x)\,dA = 0$

10. $\iint_D (2x - x)\,dA = \int_0^\pi\int_0^{\sin x} x\,dy\,dx = \int_0^\pi x\sin x\,dx$
$$= \left[-x\cos x + \sin x\right]_0^\pi = \pi$$

11. $\displaystyle\iint_D \left[\frac{\partial}{\partial x}(3x + \sin y) - \frac{\partial}{\partial y}\left(y^2 - \tan^{-1} x\right)\right] dA$
$$= \int_{-2}^2\int_{x^2}^4 (3 - 2y)\,dy\,dx$$
$$= \int_{-2}^2 \left(-4 - 3x^2 + x^4\right) dx = -\tfrac{96}{5}$$

12. The region D enclosed by C is given by $\left\{(x,y) \mid -2 \le x \le 2, -\sqrt{4 - x^2} \le y \le \sqrt{4 - x^2}\right\}$ or, in polar coordinates, $\{(r,\theta) \mid 0 \le \theta \le \pi, 0 \le r \le 2\}$. Thus,
$$\int_C xy\,dx + 2x^2\,dy = \iint_D \left[\frac{\partial}{\partial x}\left(2x^2\right) - \frac{\partial}{\partial y}(xy)\right] dA$$
$$= \iint_D (4x - x)\,dA = \int_0^\pi\int_0^2 (3r\cos\theta)\,r\,dr\,d\theta$$
$$= 3\int_0^\pi \cos\theta\,d\theta \int_0^2 r^2\,dr = 3\left[\sin\theta\right]_0^\pi \left[\tfrac{1}{3}r^3\right]_0^2$$
$$= 3(0)\left(\tfrac{8}{3}\right) = 0$$

13. $\int_C (x^3 - y^3)\,dx + (x^3 + y^3)\,dy$

$$= \iint_{1 \le x^2+y^2 \le 9} \left[\frac{\partial}{\partial x}(x^3+y^3) - \frac{\partial}{\partial y}(x^3-y^3)\right] dA$$

$$= \iint_{1 \le x^2+y^2 \le 9} (3x^2+3y^2)\,dA$$

$$= 3\int_{-\pi}^{\pi}\int_1^3 r^3\,dr\,d\theta = 6\pi\left(\tfrac{81}{4} - \tfrac{1}{4}\right) = 120\pi$$

14. The region D enclosed by C is given, in polar coordinates, by $\left\{(r,\theta) \mid 0 \le \theta \le \frac{\pi}{4}, 0 \le r \le 2\right\}$. Thus

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (y^2 - x^2 y)\,dx + xy^2\,dy$$

$$= \iint_D (y^2 - 2y + x^2)\,dA$$

$$= \int_0^{\pi/4}\int_0^2 (r^2 - 2r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^{\pi/4}\left[4 - \tfrac{16}{3}\sin\theta\right]d\theta$$

$$= \left[4\theta + \tfrac{16}{3}\cos\theta\right]_0^{\pi/4} = \pi + \tfrac{8}{3}\left(\sqrt{2} - 2\right)$$

15. $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C y^6\,dx + xy^5\,dy$

$$= \iint_D \left[\frac{\partial}{\partial x}(xy^5) - \frac{\partial}{\partial y}(y^6)\right] dA$$

$$= \iint_D -5y^5\,dA = 0$$

because $-5y^5$ is an odd function of y and D is symmetric with respect to the y-axis.

16. $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C x^3 y\,dx + x^4\,dy$

$$= \iint_{0 \le x^4+y^4 \le 1} (4x^3 - x^3)\,dA$$

$$= \int_{-1}^{1}\int_{-\sqrt[4]{1-y^4}}^{\sqrt[4]{1-y^4}} 3x^3\,dx\,dy$$

$$= \int_{-1}^{1}\left[\tfrac{3}{4}x^4\right]_{-\sqrt[4]{1-y^4}}^{\sqrt[4]{1-y^4}}\,dx = 0$$

17. $A = \oint_C x\,dy = \int_0^{2\pi} (\cos^3 t)(3\sin^2 t\cos t)\,dt$

$$= 3\int_0^{2\pi} (\cos^4 t\sin^2 t)\,dt$$

$$= 3\Big[-\tfrac{1}{6}(\sin t\cos^5 t) + \tfrac{1}{6}\left[\tfrac{1}{4}(\sin t\cos^3 t) + \tfrac{3}{8}(\cos t\sin t) + \tfrac{3}{8}t\right]\Big]_0^{2\pi}$$

$$= 3\left(\tfrac{1}{6}\right)\left(\tfrac{6}{8}\pi\right) = \tfrac{3}{8}\pi$$

Or:

$$3\int_0^{2\pi} (\cos^4 t\sin^2 t)\,dt$$

$$= 3\int_0^{2\pi} \tfrac{1}{8}\left[\tfrac{1}{2}(1 - \cos 4t) + \sin^2 2t\cos 2t\right]dt = \tfrac{3}{8}\pi$$

18. $A = \oint_C x\,dy = \int_0^{2\pi} (\cos t)(3\sin^2 t\cos t)\,dt$

$$= 3\int_0^{2\pi} \tfrac{1}{8}(1 - \cos 4t)\,dt = \tfrac{3}{4}\pi$$