13.3 The Fundamental Theorem for Line Integrals

A Click here for answers.

S Click here for solutions.

1–9 Determine whether or not $\mathbf{F}$ is a conservative vector field. If it is, find a function f such that $\mathbf{F} = \nabla f$.

1. $\mathbf{F}(x, y) = (2x - 3y)\,\mathbf{i} + (2y - 3x)\,\mathbf{j}$

2. $\mathbf{F}(x, y) = (3x^2 - 4y)\,\mathbf{i} + (4y^2 - 2x)\,\mathbf{j}$

3. $\mathbf{F}(x, y) = (x^2 + y)\,\mathbf{i} + x^2\,\mathbf{j}$

4. $\mathbf{F}(x, y) = (x^2 + y)\,\mathbf{i} + (y^2 + x)\,\mathbf{j}$

5. $\mathbf{F}(x, y) = (1 + 4x^3y^3)\,\mathbf{i} + 3x^4y^2\,\mathbf{j}$

6. $\mathbf{F}(x, y) = (y\cos x - \cos y)\,\mathbf{i} + (\sin x + x\sin y)\,\mathbf{j}$

7. $\mathbf{F}(x, y) = (e^{2x} + x\sin y)\,\mathbf{i} + x^2\cos y\,\mathbf{j}$

8. $\mathbf{F}(x, y) = (ye^{xy} + 4x^3y)\,\mathbf{i} + (xe^{xy} + x^4)\,\mathbf{j}$

9. $\mathbf{F}(x, y) = (x + y^2)\,\mathbf{i} + (2xy + y^2)\,\mathbf{j}$

10–17 (a) Find a function f such that $\mathbf{F} = \nabla f$ and (b) use part (a) to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C.

10. $\mathbf{F}(x, y) = x\,\mathbf{i} + y\,\mathbf{j}$,
C is the arc of the parabola $y = x^2$ from $(-1, 1)$ to $(3, 9)$

11. $\mathbf{F}(x, y) = y\,\mathbf{i} + x\,\mathbf{j}$,
C is the arc of the curve $y = x^4 - x^3$ from $(1, 0)$ to $(2, 8)$

12. $\mathbf{F}(x, y) = 2xy^3\,\mathbf{i} + 3x^2y^2\,\mathbf{j}$,
C: $r(t) = \sin t\,\mathbf{i} + (t^2 + 1)\,\mathbf{j}$, $0 \leqslant t \leqslant \pi/2$

13. $\mathbf{F}(x, y) = e^{2y}\,\mathbf{i} + (1 + 2xe^{2y})\,\mathbf{j}$,
C: $\mathbf{r}(t) = te^t\,\mathbf{i} + (1 + t)\,\mathbf{j}$, $0 \leqslant t \leqslant 1$

14. $\mathbf{F}(x, y, z) = y\,\mathbf{i} + (x + z)\,\mathbf{j} + y\,\mathbf{k}$,
C is the line segment from $(2, 1, 4)$ to $(8, 3, -1)$

15. $\mathbf{F}(x, y, z) = 2xy^3z^4\,\mathbf{i} + 3x^2y^2z^4\,\mathbf{j} + 4x^2y^3z^3\,\mathbf{k}$,
C: $x = t$, $y = t^2$, $z = t^3$, $0 \leqslant t \leqslant 2$

16. $\mathbf{F}(x, y, z) = (2xz + \sin y)\,\mathbf{i} + x\cos y\,\mathbf{j} + x^2\,\mathbf{k}$,
C: $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$, $0 \leqslant t \leqslant 2\pi$

17. $\mathbf{F}(x, y, z) = 4xe^z\,\mathbf{i} + \cos y\,\mathbf{j} + 2x^2e^z\,\mathbf{k}$,
C: $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^4\,\mathbf{k}$, $0 \leqslant t \leqslant 1$

18–19 Show that the line integral is independent of path and evaluate the integral.

18. $\int_C 2x\sin y\,dx + (x^2\cos y - 3y^2)\,dy$,
C is any path from $(-1, 0)$ to $(5, 1)$

19. $\int_C (2y^2 - 12x^3y^3)\,dx + (4xy - 9x^4y^2)\,dy$,
C is any path from $(1, 1)$ to $(3, 2)$

20. Find the work done by the force field

$$\mathbf{F}(x, y) = x^2y^3\,\mathbf{i} + x^3y^2\,\mathbf{j}$$

in moving an object from $P(0, 0)$ to $Q(2, 1)$.

Answers

E Click here for exercises.

S Click here for solutions.

1. $f(x, y) = x^2 - 3xy + y^2 + K$

2. Not conservative

3. Not conservative

4. $f(x, y) = \frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + K$

5. $f(x, y) = x + x^4y^3 + K$

6. $f(x, y) = y \sin x - x \cos y + K$

7. Not conservative

8. $f(x, y) = e^{xy} + x^4y + K$

9. $f(x, y) = \frac{1}{2}x^2 + xy^2 + \frac{1}{3}y^3 + K$

10. (a) $f(x, y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$
(b) 44

11. (a) $f(x, y) = xy$
(b) 16

12. (a) $f(x, y) = x^2y^3$
(b) $\frac{1}{64}\left(\pi^2 + 4\right)^3$

13. (a) $f(x, y) = xe^{2y} + y$
(b) $e^5 + 1$

14. (a) $f(x, y, z) = xy + yz$
(b) 15

15. (a) $f(x, y, z) = x^2y^3z^4$
(b) 2^{20}

16. (a) $f(x, y, z) = x^2z + x \sin y$
(b) 2π

17. (a) $f(x, y, z) = 2x^2e^z + \sin y$
(b) $2e + \sin 1$

18. $25 \sin 1 - 1$

19. -1919

20. $\frac{8}{3}$

Solutions

E Click here for exercises.

A Click here for answers.

1. $\frac{\partial}{\partial y}(2x-3y) = -3 = \frac{\partial}{\partial x}(2y-3x)$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$ which is open and simply-connected, so $\mathbf{F}$ is conservative. Thus there exists f such that $\nabla f = \mathbf{F}$, that is, $f_x(x,y) = 2x-3y$ and $f_y(x,y) = 2y-3x$. But $f_x(x,y) = 2x-3y$ implies $f(x,y) = x^2-3yx+g(y)$ and differentiating both sides of this equation with respect to y gives $f_y(x,y) = -3x+g'(y)$. Thus $2y-3x = -3x+g'(y)$ so $g'(y) = 2y$ and $g(y) = y^2+K$ where K is a constant. Hence $f(x,y) = x^2-3xy+y^2+K$ is a potential for $\mathbf{F}$.

2. $\frac{\partial}{\partial y}(3x^2-4y) = -4$, $\frac{\partial}{\partial x}(4y^2-2x) = -2$ and these are not equal, so $\mathbf{F}$ is not conservative.

3. $\frac{\partial}{\partial y}(x^2+y) = 1$, $\frac{\partial}{\partial x}(x^2) = 2x$ and these are not equal, so $\mathbf{F}$ is not conservative.

4. $\frac{\partial}{\partial y}(x^2+y) = 1 = \frac{\partial}{\partial x}(y^2+x)$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$ which is open and simply-connected. Thus $\mathbf{F}$ is conservative so there exists f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y) = x^2+y$ implies $f(x,y) = \frac{1}{3}x^3+xy+g(y)$ and differentiating both sides with respect to y gives $f_y(x,y) = x+g'(y)$. But $f_y(x,y) = y^2+x$, so $g'(y) = y^2$ or $g(y) = \frac{1}{3}y^3+K$. Hence a potential for $\mathbf{F}$ is $f(x,y) = \frac{1}{3}x^3+xy+\frac{1}{3}y^3+K$.

5. $\frac{\partial}{\partial y}(1+4x^3y^3) = 12x^3y^2 = \frac{\partial}{\partial x}(3x^4y^2)$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$ which is open and simply-connected. Thus $\mathbf{F}$ is conservative so there exists f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y) = 1+4x^3y^3$ implies $f(x,y) = x+x^4y^3+g(y)$ and $f_y(x,y) = 3x^4y^3+g'(y)$. But $f_y(x,y) = 3x^4y^2$ implies $g(y) = K$. Hence a potential for $\mathbf{F}$ is $f(x,y) = x+x^4y^3+K$.

6. $\frac{\partial}{\partial y}(y\cos x-\cos y) = \cos x+\sin y$

$$= \frac{\partial}{\partial x}(\sin x + x\sin y)$$

and the domain of $\mathbf{F}$ is $\mathbb{R}^2$ which is open and simply-connected. Thus $\mathbf{F}$ is conservative so there exists f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y) = y\cos x-\cos y$ implies $f(x,y) = y\sin x - x\cos y+g(y)$ and $f_y(x,y) = \sin x+x\sin y+g'(y)$. But $f_y(x,y) = \sin x+x\sin y$, so $g(y) = K$. Hence $f(x,y) = y\sin x-x\cos y+K$ is a potential for $\mathbf{F}$.

7. $\frac{\partial}{\partial y}(e^{2x}+x\sin y) = x\cos y$, $\frac{\partial}{\partial x}(x^2\cos y) = 2x\cos y$, so $\mathbf{F}$ is not conservative.

8. $\frac{\partial}{\partial y}(ye^{xy}+4x^3y) = e^{xy}(yx+1)+4x^3$

$$= \frac{\partial}{\partial x}(xe^{xy}+x^4)$$

and the domain of $\mathbf{F}$ is $\mathbb{R}^2$. Thus $\mathbf{F}$ is conservative so there exists f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y) = ye^{xy}+4x^3y$ implies $f(x,y) = e^{xy}+x^4y+g(y)$ and $f_y(x,y) = xe^{yx}+x^4+g'(y)$. But $f_y(x,y) = xe^{xy}+x^4$ so $g(y) = K$ and $f(x,y) = e^{xy}+x^4y+K$ is a potential for $\mathbf{F}$.

9. $\frac{\partial}{\partial y}(x+y^2) = 2y = \frac{\partial}{\partial x}(2xy+y^2)$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$. Hence $\mathbf{F}$ is conservative so there exists f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y) = x+y^2$ implies $f(x,y) = x^2/2+xy^2+g(y)$ and $f_y(x,y) = 2xy+g'(y)$. But $f_y(x,y) = 2xy+y^2$ so $g'(y) = y^2$ or $g(y) = \frac{1}{3}y^3+K$. Then $f(x,y) = \frac{1}{2}x^2+xy^2+\frac{1}{3}y^3+K$ is a potential for $\mathbf{F}$.

10. (a) $f_x(x,y) = x$ implies $f(x,y) = \frac{1}{2}x^2+g(y)$ and $f_y(x,y) = g'(y)$. But $f_y(x,y) = y$ so $g(y) = \frac{1}{2}y^2+K$ and $f(x,y) = \frac{1}{2}x^2+\frac{1}{2}y^2+K$ (or set $K=0$.)

(b) $\int_C \mathbf{F}\cdot d\mathbf{r} = f(3,9)-f(-1,1) = 44$

11. (a) $f_x(x,y) = y$ implies $f(x,y) = xy+g(y)$ and $f_y(x,y) = x+g'(y)$. But $f_y(x,y) = x$ so $f(x,y) = xy$ (setting $K=0$).

(b) $\int_C \mathbf{F}\cdot d\mathbf{r} = f(2,8)-f(1,0) = 16$

12. (a) $f_x(x,y) = 2xy^3$ implies $f(x,y) = x^2y^3+g(y)$ and $f_y(x,y) = 3x^2y^2+g'(y)$. But $f_y(x,y) = 3x^2y^2$ so $f(x,y) = x^2y^3$ (setting $K=0$).

(b) Since $\mathbf{r}(0) = \langle 0,1\rangle$ and $\mathbf{r}\left(\frac{\pi}{2}\right) = \left\langle 1, \frac{1}{4}\left(\pi^2+4\right)\right\rangle$, $\int_C \mathbf{F}\cdot d\mathbf{r} = f\left(1,\frac{1}{4}\left(\pi^2+4\right)\right) - f(0,1) = \frac{1}{64}\left(\pi^2+4\right)^3$.

13. (a) $f_x(x,y) = e^{2y}$ implies $f(x,y) = xe^{2y}+g(y)$ and $f_y(x,y) = 2xe^{2y}+g'(y)$. But $f_y(x,y) = 1+2xe^{2y}$ so $g'(y) = 1$ and $g(y) = y$ (setting $K=0$). Thus $f(x,y) = xe^{2y}+y$.

(b) Since $\mathbf{r}(0) = \langle 0,1\rangle$ and $\mathbf{r}(1) = \langle e,2\rangle$, $\int_C \mathbf{F}\cdot d\mathbf{r} = f(e,2)-f(0,1) = (e)e^4+2-1 = e^5+1$.

14. (a) $f_x(x,y,z) = y$ implies $f(x,y,z) = xy + g(y,z)$ and $f_y(x,y,z) = x + \partial g/\partial y$. But $f_y(x,y,z) = x + z$ so $\partial g/\partial y = z$ and $g(y,z) = yz + h(z)$. Thus $f(x,y,z) = xy + yz + h(z)$ and $f_z(x,y,z) = y + h'(z)$. But $f_z(x,y,z) = y$ so $h'(z) = 0$ or $h(z) = K$. Hence $f(x,y,z) = xy + yz$ (setting $K = 0$).

(b) $\int_C \mathbf{F} \cdot d\mathbf{r} = f(8,3,-1) - f(2,1,4) = 21 - 6 = 15$

15. (a) $f_x(x,y,z) = 2xy^3z^4$ implies $f(x,y,z) = x^2y^3z^4 + g(y,z)$ and $f_y(x,y,z) = 3x^2y^2z^4 + g_y(y,z)$. But $f_y(x,y,z) = 3x^2y^2z^4$, so $g_y(y,z) = h(z)$, and also $f(x,y,z) = x^2y^3z^4 + h(z)$, implying $f_z(x,y,z) = 4x^2y^3z^3 + h'(z)$. But $f_z(x,y,z) = 4x^2y^3z^3$, so $h'(z) = 0$. Hence $f(x,y,z) = x^2y^3z^4$.

(b) $\mathbf{r}(0) = \langle 0,0,0\rangle$ and $\mathbf{r}(2) = \langle 2,4,8\rangle$ so $\int_C \mathbf{F} \cdot d\mathbf{r} = f(2,4,8) - f(0,0,0) = 2^2 \cdot 4^3 \cdot 8^4 = 2^{20}$.

16. (a) $f_x(x,y,z) = 2xz + \sin y$ implies $f(x,y,z) = x^2z + x\sin y + g(y,z)$ and $f_y(x,y,z) = x\cos y + g_y(y,z)$. But $f_y(x,y,z) = x\cos y$ so $g_y(y,z) = 0$ and $f(x,y,z) = x^2z + x\sin y + h(z)$. Thus $f_z(x,y,z) = x^2 + h'(z)$. But $f_z(x,y,z) = x^2$ so $h'(z) = 0$ and $f(x,y,z) = x^2z + x\sin y$ (setting $K = 0$).

(b) $\mathbf{r}(0) = \langle 1,0,0\rangle$, $\mathbf{r}(2\pi) = \langle 1,0,2\pi\rangle$. Thus $\int_C \mathbf{F} \cdot d\mathbf{r} = f(1,0,2\pi) - f(1,0,0) = 2\pi$.

17. (a) $f_x(x,y,z) = 4xe^z$ implies $f(x,y,z) = 2x^2e^z + g(y,z)$ and $f_y(x,y,z) = g_y(y,z)$. But $f_y(x,y,z) = \cos y$ so $g_y(y,z) = \cos y$ or $g(y,z) = \sin y + h(z)$. Thus $f(x,y,z) = 2x^2e^z + \sin y + h(z)$, and $f_z(x,y,z) = 2x^2e^z + h'(z)$. But $f_z(x,y,z) = 2x^2e^z$ so $h'(z) = 0$ and $f(x,y,z) = 2x^2e^z + \sin y$ (setting $K = 0$).

(b) $\mathbf{r}(0) = \langle 0,0,0\rangle$, $\mathbf{r}(1) = \langle 1,1,1\rangle$ so $\int_C \mathbf{F} \cdot d\mathbf{r} = f(1,1,1) - f(0,0,0) = 2e + \sin 1$.

18. Here $\mathbf{F}(x,y) = (2x\sin y)\,\mathbf{i} + \left(x^2\cos y - 3y^2\right)\mathbf{j}$. Then $f(x,y) = x^2\sin y - y^3$ is a potential function for $\mathbf{F}$, that is, $\nabla f = \mathbf{F}$ so $\mathbf{F}$ is conservative and thus its line integral is independent of path. Hence

$$\int_C 2x\sin y\,dx + \left(x^2\cos y - 3y^2\right)dy = \int_C \mathbf{F} \cdot d\mathbf{r} = f(5,1) - f(-1,0) = 25\sin 1 - 1$$

19. Here $\mathbf{F}(x,y) = \left(2y^2 - 12x^3y^3\right)\mathbf{i} + \left(4xy - 9x^4y^2\right)\mathbf{j}$. Then $f(x,y) = 2xy^2 - 3x^4y^3$ is a potential function for $\mathbf{F}$, that is, $\nabla f = \mathbf{F}$. Hence $\mathbf{F}$ is conservative and its line integral is independent of path.

$$\int_C \left(2y^2 - 12x^3y^3\right)dx + \left(4xy - 9x^4y^2\right)dy = \int_C \mathbf{F} \cdot d\mathbf{r} = f(3,2) - f(1,1) = -1920 - (-1) = -1919$$

20. $\mathbf{F}(x,y) = x^2y^3\,\mathbf{i} + x^3y^2\,\mathbf{j}$, $W = \int_C \mathbf{F} \cdot d\mathbf{r}$. Since $\dfrac{\partial}{\partial y}\left(x^2y^3\right) = 3x^2y^2 = \dfrac{\partial}{\partial x}\left(x^3y^2\right)$, there exists a function f such that $\nabla f = \mathbf{F}$. In fact, $f_x = x^2y^3 \Rightarrow f(x,y) = \frac{1}{3}x^3y^3 + g(y) \Rightarrow f_y = x^3y^2 + g'(y) \Rightarrow g'(y) = 0$, so we can take $f(x,y) = \frac{1}{3}x^3y^3$. Thus $W = \int_C \mathbf{F} \cdot d\mathbf{r} = f(2,1) - f(0,0) = \frac{1}{3}\left(2^3\right)\left(1^3\right) - 0 = \frac{8}{3}$.