13.1 Vector Fields

A Click here for answers.

S Click here for solutions.

1–5 Sketch the vector field **F** by drawing a diagram like Figure 5 or Figure 9.

1. $\mathbf{F}(x, y) = x\,\mathbf{i} + y\,\mathbf{j}$

2. $\mathbf{F}(x, y) = x\,\mathbf{i} - y\,\mathbf{j}$

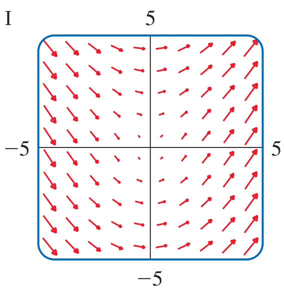
3. $\mathbf{F}(x, y) = y\,\mathbf{i} + \mathbf{j}$

4. $\mathbf{F}(x, y) = -x\,\mathbf{i} + 2y\,\mathbf{j}$

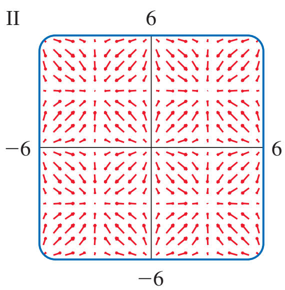
5. $\mathbf{F}(x, y, z) = \mathbf{j} + \mathbf{k}$

6–8 Match the vector fields **F** with the plots labeled I–III. Give reasons for your choices.

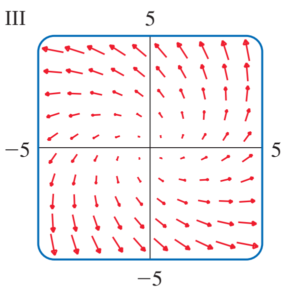
I



II



III



6. $\mathbf{F}(x, y) = \langle 2x - 3y, 2x + 3y \rangle$

7. $\mathbf{F}(x, y) = \langle \sin x, \sin y \rangle$

8. $\mathbf{F}(x, y) = \langle \ln(1 + x^2 + y^2), x \rangle$

9–14 Find the gradient vector field of f.

9. $f(x, y) = x^5 - 4x^2y^3$

10. $f(x, y) = \sin(2x + 3y)$

11. $f(x, y) = e^{3x} \cos 4y$

12. $f(x, y, z) = xyz$

13. $f(x, y, z) = xy^2 - yz^3$

14. $f(x, y, z) = x \ln(y - z)$

15–16 Find the gradient vector field ∇f of f and sketch it.

15. $f(x, y) = x^2 - \frac{1}{2}y^2$

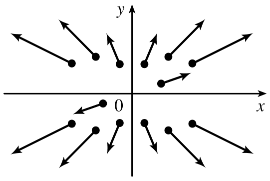
16. $f(x, y) = \ln \sqrt{x^2 + y^2}$

Answers

E Click here for exercises.

S Click here for solutions.

1.



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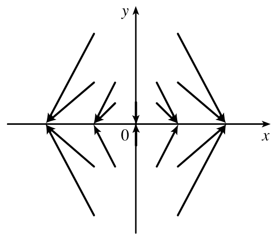
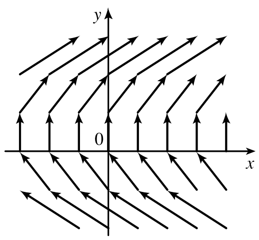


3.



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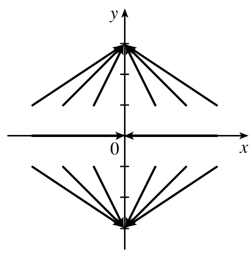
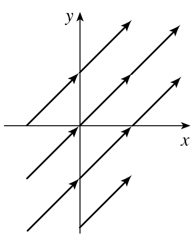


5.



6. III **7.** II **8.** I

9. $\left(5x^4 - 8xy^3\right)\mathbf{i} - \left(12x^2y^2\right)\mathbf{j}$

10. $2\cos(2x+3y)\,\mathbf{i} + 3\cos(2x+3y)\,\mathbf{j}$

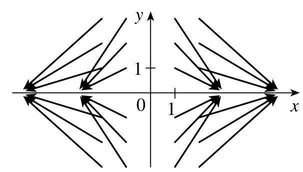
11. $\langle 3e^{3x}\cos 4y, -4e^{3x}\sin 4y\rangle$

12. $\langle yz, xz, xy\rangle$

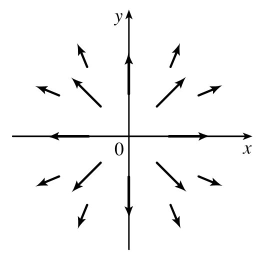
13. $\langle y^2, 2xy - z^3, -3yz^2\rangle$

14. $\left\langle \ln(y-z), \dfrac{x}{y-z}, -\dfrac{x}{y-z}\right\rangle$

15. $2x\,\mathbf{i} - y\,\mathbf{j}$



16. $\dfrac{x\,\mathbf{i} + y\,\mathbf{j}}{x^2+y^2}$

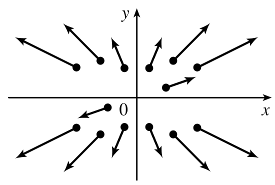


Solutions

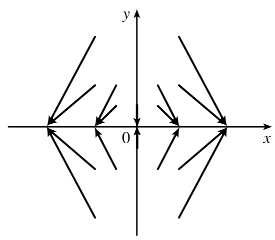
E Click here for exercises.

A Click here for answers.

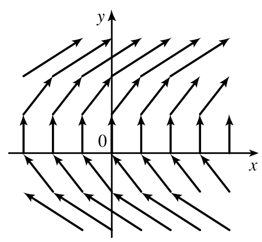
1. $\mathbf{F}(x,y) = x\,\mathbf{i} + y\,\mathbf{j}$. The length of the vector $x\,\mathbf{i} + y\,\mathbf{j}$ is the distance from $(0,0)$ to (x,y). Each vector points away from the origin.



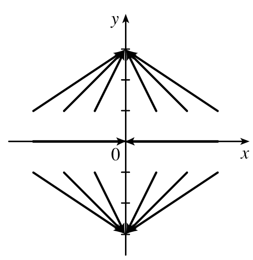
2. $\mathbf{F}(x,y) = x\,\mathbf{i} - y\,\mathbf{j}$. The length of the vector $x\,\mathbf{i} - y\,\mathbf{j}$ is the distance from $(0,0)$ to (x,y). For each (x,y), $\mathbf{F}(x,y)$ terminates on the x-axis at the point $(2x,0)$.



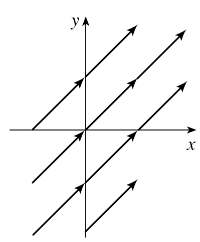
3. $\mathbf{F}(x,y) = y\,\mathbf{i} + \mathbf{j}$. The length of the vector $y\,\mathbf{i} + \mathbf{j}$ is $\sqrt{y^2+1}$. Vectors are tangent to parabolas opening about the x-axis.



4. $\mathbf{F}(x,y) = -x\,\mathbf{i} + 2y\,\mathbf{j}$. The length of the vector $-x\,\mathbf{i} + 2y\,\mathbf{j}$ is $\sqrt{x^2+4y^2}$. $\mathbf{F}(x,y)$ terminates on the y-axis at the point $(0,3y)$.



5. $\mathbf{F}(x,y,z) = \mathbf{j} + \mathbf{k}$. The length of $\mathbf{F}(x,y,z)$ is $\sqrt{2}$. The graph is shown in the yz-plane because in parallel planes $x = a$, the graph is identical to this.



6. $\mathbf{F}(x,y) = \langle 2x-3y, 2x+3y \rangle$ corresponds to graph III, since as we move to the right (so x increases and y is constant), both the x- and the y-components of the vectors get larger, and as we move upward (so y increases and x is constant), the x-components decrease, while the y-components increase.

7. $\mathbf{F}(x,y) = \langle \sin x, \sin y \rangle$ corresponds to graph II, since the vector field is the same on each square of the form $[2n\pi, 2(n+1)\pi] \times [2m\pi, 2(m+1)\pi]$, m, n any integers.

8. $\mathbf{F}(x,y) = \langle \ln(1+x^2+y^2), x \rangle$ corresponds to graph I, since $\ln(1+x^2+y^2)$ is always positive, so all vectors point to the right.

9.
$$\begin{aligned}\nabla f(x,y) &= f_x(x,y)\,\mathbf{i} + f_y(x,y)\,\mathbf{j} \\ &= (5x^4 - 8xy^3)\,\mathbf{i} - (12x^2y^2)\,\mathbf{j}\end{aligned}$$

10.
$$\begin{aligned}\nabla f(x,y) &= f_x(x,y)\,\mathbf{i} + f_y(x,y)\,\mathbf{j} \\ &= 2\cos(2x+3y)\,\mathbf{i} + 3\cos(2x+3y)\,\mathbf{j}\end{aligned}$$

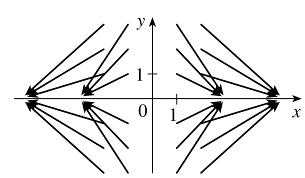
11. $\nabla f(x,y) = \langle f_x, f_y \rangle = \langle 3e^{3x}\cos 4y, -4e^{3x}\sin 4y \rangle$

12. $\nabla f(x,y,z) = \langle f_x, f_y, f_z \rangle = \langle yz, xz, xy \rangle$

13. $\nabla f(x,y,z) = \langle f_x, f_y, f_z \rangle = \langle y^2, 2xy - z^3, -3yz^2 \rangle$

14.
$$\begin{aligned}\nabla f(x,y,z) &= \langle f_x, f_y, f_z \rangle \\ &= \left\langle \ln(y-z), \frac{x}{y-z}, -\frac{x}{y-z} \right\rangle\end{aligned}$$

15. $f(x,y) = x^2 - \frac{1}{2}y^2$, $\nabla f(x,y) = 2x\,\mathbf{i} - y\,\mathbf{j}$. The length of $\nabla f(x,y)$ is $\sqrt{4x^2+y^2}$, and $\nabla f(x,y)$ terminates on the x-axis at the point $(3x,0)$.



16. $f(x,y) = \ln\sqrt{x^2+y^2} = \frac{1}{2}\ln(x^2+y^2) \quad \Rightarrow$

$$\nabla f = \tfrac{1}{2}\nabla \ln(x^2+y^2) = \frac{x}{x^2+y^2}\,\mathbf{i} + \frac{y}{x^2+y^2}\,\mathbf{j} = \frac{x\,\mathbf{i} + y\,\mathbf{j}}{x^2+y^2}$$

The length of ∇f decreases as x and/or y increase and all the vectors "flow out" away from the origin.