12.9 Change of Variables in Multiple Integrals

A Click here for answers.

S Click here for solutions.

1–6 Find the Jacobian of the transformation.

1. $x = u - 2v, \quad y = 2u - v$

2. $x = u - v^2, \quad y = u + v^2$

3. $x = e^{2u} \cos v, \quad y = e^{2u} \sin v$

4. $x = se^t, \quad y = se^{-t}$

5. $x = u + v + w, \quad y = u + v - w, \quad z = u - v + w$

6. $x = 2u, \quad y = 3v^2, \quad z = 4w^3$

7–8 Find the image of the set S under the given transformation.

7. $S = \{(u, v) \mid 0 \leqslant u \leqslant 2,\ 0 \leqslant v \leqslant 1\}$;
$x = u - 2v,\ y = 2u - v$

8. $S = \{(u, v) \mid 0 \leqslant u \leqslant 1,\ u \leqslant v \leqslant 1\}$;
$x = u^2,\ y = v$

9–10 Use the given transformation to evaluate the integral.

9. $\iint_R (3x + 4y)\, dA$, where R is the region bounded by the lines $y = x$, $y = x - 2$, $y = -2x$, and $y = 3 - 2x$;
$x = \frac{1}{3}(u + v),\ y = \frac{1}{3}(v - 2u)$

10. $\iint_R (x + y)\, dA$, where R is the square with vertices (0, 0), (2, 3), (5, 1), and (3, −2); $\quad x = 2u + 3v,\ y = 3u - 2v$

11–12 Evaluate the integral by making an appropriate change of variables.

11. $\iint_R xy\, dA$, where R is the region bounded by the lines $2x - y = 1$, $2x - y = -3$, $3x + y = 1$, and $3x + y = -2$

12. $\displaystyle\iint_R \frac{x + 2y}{\cos(x - y)}\, dA$, where R is the parallelogram bounded by the lines $y = x$, $y = x - 1$, $x + 2y = 0$, and $x + 2y = 2$

Answers

E Click here for exercises.

S Click here for solutions.

1. 3
2. $4v$
3. $2e^{4u}$
4. $-2s$
5. -4
6. $144vw^2$
7. The parallelogram bounded by the lines $y = 2x$, $y = 2x + 3$, $x = 2y$, $x = 2y - 6$
8. The figure bounded by the lines $x = 0$, $y = 1$ and the parabola $x = y^2$
9. $\frac{11}{3}$
10. 39
11. $-\frac{66}{125}$
12. $\frac{2}{3} \ln(\sec 1 + \tan 1)$

Solutions

E Click here for exercises.

A Click here for answers.

1. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix}$
$= 1(-1) - 2(-2) = 3$

2. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 1 & -2v \\ 1 & 2v \end{vmatrix}$
$= 2v - (-2v) = 4v$

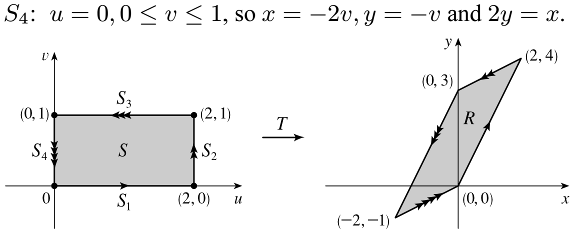
3. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 2e^{2u}\cos v & -e^{2u}\sin v \\ 2e^{2u}\sin v & e^{2u}\cos v \end{vmatrix}$
$= 2e^{4u}\left(\cos^2 v + \sin^2 v\right) = 2e^{4u}$

4. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} e^t & se^t \\ e^{-t} & -se^{-t} \end{vmatrix}$
$= -s - s = -2s$

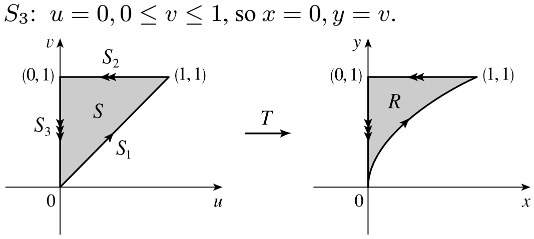
5. $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$
$= 1(1-1) - 1(1+1) + 1(-1-1) = -4$

6. $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 6v & 0 \\ 0 & 0 & 12w^2 \end{vmatrix}$
$= (2)(6v)\left(12w^2\right) = 144vw^2$

7. S_1: $v = 0, 0 \le u \le 2$, so $x = u, y = 2u$ and $y = 2x$.
S_2: $u = 2, 0 \le v \le 1$, so $x = 2 - 2v, y = 4 - v$ and $x = 2y - 6$.
S_3: $v = 1, 0 \le u \le 2$, so $x = u - 2, y = 2u - 1$ and $y = 2x + 3$.
S_4: $u = 0, 0 \le v \le 1$, so $x = -2v, y = -v$ and $2y = x$.



8. S_1: $u = v, 0 \le u \le 1$, so $x = u^2, y = u$, and $x = y^2$.
S_2: $v = 1, 0 \le u \le 1$, so $x = u^2, y = 1$.
S_3: $u = 0, 0 \le v \le 1$, so $x = 0, y = v$.



9. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{vmatrix} = \dfrac{1}{3}$ and
$3x + 4y = (u+v) + \frac{4}{3}(v - 2u) = \frac{1}{3}(7v - 5u)$. To find the region S in the uv-plane that corresponds to R we first find the corresponding boundary under the given tranformation: The line $y = x$ is the image of
$\frac{1}{3}(v - 2u) = \frac{1}{3}(u + v)$ or $u = 0$, $y = x - 2 \Rightarrow$
$\frac{1}{3}(v - 2u) = \frac{1}{3}(u + v) - 2$ or $u = 2$, $y = -2x \Rightarrow$
$\frac{1}{3}(v - 2u) = -\frac{2}{3}(u + v)$ or $v = 0$, and $y = 3 - 2x \Rightarrow$
$\frac{1}{3}(v - 2u) = 3 - \frac{2}{3}(u + v)$ or $v = 3$. Thus S is the rectangle $[0,2] \times [0,3]$ in the uv-plane and

$$\iint_R (3x + 4y)\,dA = \iint_S \frac{1}{3}(7v - 5u)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$
$$= \int_0^3 \int_0^2 \tfrac{1}{3}(7v - 5u)\left(\tfrac{1}{3}\right) du\,dv$$
$$= \tfrac{1}{9}\int_0^3 (14v - 10)\,dv = \tfrac{1}{9}(33) = \tfrac{11}{3}$$

10. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix} = -13$, $x + y = 5u + v$ and since
$u = \dfrac{2x + 3y}{13}$ and $v = \dfrac{3x - 2y}{13}$, R is the image of the square with vertices $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$. Thus
$\iint_R (x + y)\,dA = \int_0^1 \int_0^1 (5u + v)\,|-13|\,du\,dv$
$= 13\int_0^1 \left(\frac{5}{2} + v\right) dv = 13(3) = 39$

11. Letting $u = 2x - y$ and $v = 3x + y$, we have $x = \frac{1}{5}(u + v)$,
$y = \frac{1}{5}(2v - 3u)$. Then $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{5} & \frac{1}{5} \\ -\frac{3}{5} & \frac{2}{5} \end{vmatrix} = \dfrac{1}{5}$ and

$$\iint_R xy\,dA = \int_{-2}^{1}\int_{-3}^{1} \frac{(u+v)(2v-3u)}{25}\left(\frac{1}{5}\right) du\,dv$$
$$= \tfrac{1}{125}\int_{-2}^{1}\int_{-3}^{1}\left(2v^2 - uv - 3u^2\right) du\,dv$$
$$= \tfrac{1}{125}\int_{-2}^{1}\left(8v^2 + 4v - 28\right) dv = -\tfrac{66}{125}$$

12. Let $u = x - y$, $v = x + 2y$, so $y = \frac{1}{3}(v - u)$ and
$x = \frac{1}{3}(2u + v)$. Then $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{vmatrix} = \dfrac{1}{3}$ and

$$\iint_R \frac{x + 2y}{\cos(x - y)}\,dA = \frac{1}{3}\int_0^1\int_0^2 \frac{v}{\cos u}\,dv\,du$$
$$= \tfrac{2}{3}\int_0^1 \sec u\,du = \tfrac{2}{3}\left[\ln|\sec u + \tan u|\right]_0^1$$
$$= \tfrac{2}{3}\left[\ln(\sec 1 + \tan 1) - \ln 1\right] = \tfrac{2}{3}\ln(\sec 1 + \tan 1)$$