12.7 Triple Integrals

A Click here for answers.

S Click here for solutions.

1. Evaluate the integral $\iiint_E (x^2 + yz)\, dV$, where

$$E = \{(x, y, z) \mid 0 \leqslant x \leqslant 2,\ -3 \leqslant y \leqslant 0,\ -1 \leqslant z \leqslant 1\}$$

using three different orders of integration.

2–6 Evaluate the iterated integral.

2. $\int_0^1 \int_0^z \int_0^y xyz\, dx\, dy\, dz$

3. $\int_0^1 \int_x^{2x} \int_0^{x+y} 2xy\, dz\, dy\, dx$

4. $\int_0^{\pi} \int_0^2 \int_0^{\sqrt{4-z^2}} z \sin y\, dx\, dz\, dy$

5. $\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^x yz\, dy\, dz\, dx$

6. $\int_1^2 \int_0^x \int_0^{1-y} x^3y^2z\, dz\, dy\, dx$

7–11 Evaluate the triple integral.

7. $\iiint_E yz\, dV$, where
$E = \{(x, y, z) \mid 0 \leqslant z \leqslant 1, 0 \leqslant y \leqslant 2z, 0 \leqslant x \leqslant z + 2\}$

8. $\iiint_E e^x\, dV$, where
$E = \{(x, y, z) \mid 0 \leqslant y \leqslant 1, 0 \leqslant x \leqslant y, 0 \leqslant z \leqslant x + y\}$

9. $\iiint_E y\, dV$, where E lies under the plane $z = x + 2y$ and above the region in the xy-plane bounded by the curves $y = x^2$, $y = 0$, and $x = 1$

10. $\iiint_E x\, dV$, where E is bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $3x + 2y + z = 6$

11. $\iiint_E z\, dV$, where E is bounded by the planes $x = 0$, $y = 0$, $z = 0$, $y + z = 1$, and $x + z = 1$

12–15 Use a triple integral to find the volume of the given solid.

12. The tetrahedron bounded by the coordinate planes and the plane $2x + 3y + 6z = 12$

13. The solid bounded by the elliptic cylinder $4x^2 + z^2 = 4$ and the planes $y = 0$ and $y = z + 2$

14. The solid bounded by the cylinder $x = y^2$ and the planes $z = 0$ and $x + z = 1$

15. The solid enclosed by the paraboloids $z = x^2 + y^2$ and $z = 18 - x^2 - y^2$

CAS **16.** Evaluate the triple integral exactly:

$$\int_0^2 \int_{-1}^{\sin x} \int_{z-x}^{z+x} e^{3x}(5y + 2z)\, dy\, dz\, dx$$

17. Set up, but do not evaluate, integral expressions for (a) the mass, (b) the center of mass, and (c) the moment of inertia about the z-axis of the solid bounded by the paraboloid $x = 4y^2 + 4z^2$ and the plane $x = 4$ with density function $\rho(x, y, z) = x^2 + y^2 + z^2$.

18. Find the average value of the function $f(x, y, z) = x + y + z$ over the tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

Answers

E Click here for exercises.

S Click here for solutions.

1. 16

2. $\frac{1}{48}$

3. $\frac{23}{15}$

4. $\frac{16}{3}$

5. $\frac{162}{5}$

6. $\frac{7387}{10{,}080}$

7. $\frac{7}{5}$

8. $\frac{1}{2}(7-2e)$

9. $\frac{5}{28}$

10. 3

11. $\frac{1}{12}$

12. 8

13. 4π

14. $\frac{8}{15}$

15. 81π

16. $e^6\left(-\frac{35}{18}-\frac{511}{338}\cos 4-\frac{140}{169}\sin 4\right)-\frac{749}{1521}$

17. (a) $\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{4y^2+4z^2}^{4}\left(x^2+y^2+z^2\right)dx\,dz\,dy$

(b) $(\overline{x},\overline{y},\overline{z})$ where

$$\overline{x}=\frac{1}{m}\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{4y^2+4z^2}^{4}x\left(x^2+y^2+z^2\right)dx\,dz\,dy$$

$$\overline{y}=\frac{1}{m}\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{4y^2+4z^2}^{4}y\left(x^2+y^2+z^2\right)dx\,dz\,dy$$

$$\overline{z}=\frac{1}{m}\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{4y^2+4z^2}^{4}z\left(x^2+y^2+z^2\right)dx\,dz\,dy$$

(c) $\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{4y^2+4z^2}^{4}\left(x^2+y^2\right)\left(x^2+y^2+z^2\right)dx\,dz\,dy$

18. $\frac{3}{4}$

Solutions

E Click here for exercises.

A Click here for answers.

1. $\int_0^2\int_{-3}^0\int_{-1}^1 \left(x^2+yz\right) dz\,dy\,dx$
$= \int_0^2\int_{-3}^0 \left[x^2z+\frac{1}{2}yz^2\right]_{z=-1}^{z=1} dy\,dx = \int_0^2\int_{-3}^0 2x^2\,dy\,dx$
$= \int_0^2 \left[2x^2y\right]_{y=-3}^{y=0} dx = \int_0^2 6x^2\,dx = 2x^3\Big]_0^2 = 16$

$\int_{-1}^1\int_{-3}^0\int_0^2 \left(x^2+yz\right) dx\,dy\,dz$
$= \int_{-1}^1\int_{-3}^0 \left[\frac{1}{3}x^3+xyz\right]_{x=0}^{x=2} dy\,dz$
$= \int_{-1}^1\int_{-3}^0 \left(\frac{8}{3}+2yz\right) dy\,dz = \int_{-1}^1 \left[\frac{8}{3}y+y^2z\right]_{y=-3}^{y=0} dz$
$= \int_{-1}^1 (8-9z)\,dz = \left[8z-\frac{9}{2}z^2\right]_{-1}^1 = 16$

$\int_{-1}^1\int_0^2\int_{-3}^0 \left(x^2+yz\right) dy\,dx\,dz$
$= \int_{-1}^1\int_0^2 \left[x^2y+\frac{1}{2}y^2z\right]_{y=-3}^{y=0} dx\,dz$
$= \int_{-1}^1\int_0^2 \left(3x^2-\frac{9}{2}z\right) dx\,dz = \int_{-1}^1 \left[x^3-\frac{9}{2}xz\right]_{x=0}^{x=2} dz$
$= \int_{-1}^1 (8-9z)\,dz = \left[8z-\frac{9}{2}z^2\right]_{-1}^1 = 16$

2. $\int_0^1\int_0^z\int_0^y xyz\,dx\,dy\,dz = \int_0^1\int_0^z \left(\frac{1}{2}y^3z\right) dy\,dz$
$= \int_0^1 \frac{1}{8}z^5\,dz = \frac{1}{48}z^6\Big|_0^1 = \frac{1}{48}$

3. $\int_0^1\int_x^{2x}\int_0^{x+y} 2xy\,dz\,dy\,dx$
$= \int_0^1\int_x^{2x} \left(2x^2y+2xy^2\right) dy\,dx = \int_0^1 \left[x^2y^2+\frac{2}{3}xy^3\right]_x^{2x} dx$
$= \int_0^1 \frac{23}{3}x^4\,dx = \frac{23}{15}x^4\Big|_0^1 = \frac{23}{15}$

4. $\int_0^\pi\int_0^2\int_0^{\sqrt{4-z^2}} z\sin y\,dx\,dz\,dy$
$= \int_0^\pi\int_0^2 z\sqrt{4-z^2}\sin y\,dz\,dy$
$= \int_0^\pi \left[-\frac{1}{3}\left(4-z^2\right)^{3/2}\right]_0^2 \sin y\,dy = \int_0^\pi \frac{8}{3}\sin y\,dy$
$= -\frac{8}{3}\cos y\Big|_0^\pi = \frac{16}{3}$

5. $\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^x yz\,dy\,dz\,dx = \int_0^3\int_0^{\sqrt{9-x^2}} \left(\frac{1}{2}x^2z\right) dz\,dx$
$= \int_0^3 \left[\frac{1}{4}x^2z^2\right]_0^{\sqrt{9-x^2}} dx = \int_0^3 \frac{1}{4}\left(9x^2-x^4\right) dx$
$= \left[3x^3-\frac{1}{5}x^5\right]_0^3 = \frac{162}{5}$

6. $\int_1^2\int_0^x\int_0^{1-y} x^3y^2z\,dz\,dy\,dx = \int_1^2\int_0^x \left[\frac{1}{2}x^3y^2z^2\right]_{z=0}^{z=1-y} dy\,dx$
$= \int_1^2\int_0^x \frac{1}{2}x^3y^2(1-y)^2\,dy\,dx$
$= \int_1^2\int_0^x \left(\frac{1}{2}x^3y^2-x^3y^3+\frac{1}{2}x^3y^4\right) dy\,dx$
$= \int_1^2 \left[\frac{1}{6}x^3y^3-\frac{1}{4}x^3y^4+\frac{1}{10}x^3y^5\right]_{y=0}^{y=x} dx$
$= \int_1^2 \left(\frac{1}{6}x^6-\frac{1}{4}x^7+\frac{1}{10}x^8\right) dx$
$= \left[\frac{1}{42}x^7-\frac{1}{32}x^8+\frac{1}{90}x^9\right]_1^2$
$= \frac{128}{42}-\frac{256}{32}+\frac{512}{90}-\frac{1}{42}+\frac{1}{32}-\frac{1}{90} = \frac{7387}{10{,}080}$

7. $\int_0^1\int_0^{2z}\int_0^{z+2} yz\,dx\,dy\,dz = \int_0^1\int_0^{2z} yz\,(z+2)\,dy\,dz$
$= \int_0^1 \left(2z^4+4z^3\right) dz = \frac{7}{5}$

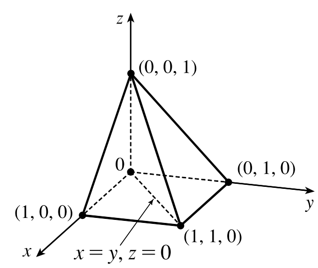
8. $\int_0^1\int_0^y\int_0^{x+y} e^x\,dz\,dx\,dy = \int_0^1\int_0^y (x+y)\,e^x\,dx\,dy$
$= \int_0^1 \left[(x+y-1)\,e^x\right]_0^y dy = \int_0^1 \left[(2y-1)\,e^y-(y-1)\right] dy$
$= \left[2ye^y-3e^y-\frac{1}{2}y^2+y\right]_0^1 = \frac{1}{2}(7-2e)$

9. $\int_0^1\int_0^{x^2}\int_0^{x+2y} y\,dz\,dy\,dx = \int_0^1\int_0^{x^2} \left(yx+2y^2\right) dy\,dx$
$= \int_0^1 \left[\frac{1}{2}xy^2+\frac{2}{3}y^3\right]_0^{x^2} dx = \int_0^1 \left(\frac{1}{2}x^5+\frac{2}{3}x^6\right) dx$
$= \left[\frac{1}{12}x^6+\frac{2}{21}x^7\right]_0^1 = \frac{5}{28}$

10. Here E is the region that lies under the plane $3x+2y+z=6$ and above the region in the xy-plane bounded by the lines $x=0$, $y=0$ and $3x+2y=6$, so

$\iiint_E x\,dV = \int_0^2\int_0^{3-3x/2}\int_0^{6-3x-2y} x\,dz\,dy\,dx$
$= \int_0^2\int_0^{3-3x/2} \left(6x-3x^2-2xy\right) dy\,dx$
$= \int_0^2 \left[\left(6x-3x^2\right)\left(3-\frac{3}{2}x\right)-x\left(3-\frac{3}{2}x\right)^2\right] dx$
$= 9\int_0^2 \left(x-x^2+\frac{1}{4}x^3\right) dx$
$= 9\left[\frac{1}{2}x^2-\frac{1}{3}x^3+\frac{1}{16}x^4\right]_0^2 = 3$

11.



By symmetry $\iiint_E z\,dV = 2\iiint_{E'} z\,dV$ where E' is the part of E to the left [as viewed from $(10,10,0)$] of the plane $x=y$. So

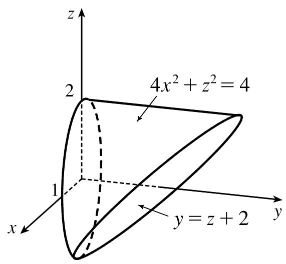
$\iiint_E z\,dV = \int_0^1\int_y^1\int_0^{1-x} 2z\,dz\,dx\,dy = \int_0^1\int_y^1 (1-x)^2\,dx\,dy$
$= \int_0^1 \left[-\frac{1}{3}(1-x)^3\right]_{x=y}^{x=1} dy = \int_0^1 \frac{1}{3}(1-y)^3\,dy$
$= \frac{1}{12}(1-y)^4\Big]_0^1 = \frac{1}{12}$

12. The plane $2x+3y+6z=12$ intersects the xy-plane when $2x+3y+6(0)=12 \;\Rightarrow\; y=4-\frac{2}{3}x$. So
$E = \left\{(x,y,z) \mid 0\le x\le 6,\ 0\le y\le 4-\frac{2}{3}x,\ 0\le z\le \frac{1}{6}(12-2x-3y)\right\}$

and

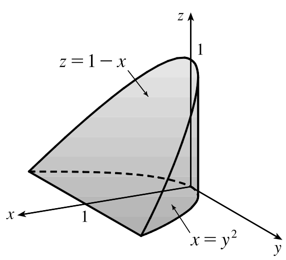
$V = \int_0^6\int_0^{4-2x/3}\int_0^{(12-2x-3y)/6} dz\,dy\,dx$
$= \frac{1}{6}\int_0^6\int_0^{4-2x/3} (12-2x-3y)\,dy\,dx$
$= \frac{1}{6}\int_0^6 \left[12y-2xy-\frac{3}{2}y^2\right]_{y=0}^{y=4-2x/3} dx$
$= \frac{1}{6}\int_0^6 \left[\frac{(12-2x)^2}{3}-\frac{3}{2}\frac{12-2x}{9}\right] dx$
$= \frac{1}{36}\int_0^6 (12-2x)^2\,dx = \left[\frac{1}{36}\left(-\frac{1}{6}\right)(12-2x)^3\right]_0^6 = 8$

13.



$$V=\int_{-1}^{1}\int_{-\sqrt{4-4x^2}}^{\sqrt{4-4x^2}}\int_{0}^{z+2}dy\,dz\,dx$$
$$=2\int_{0}^{1}\int_{-\sqrt{4-4x^2}}^{\sqrt{4-4x^2}}\int_{0}^{z+2}dy\,dz\,dx$$
$$=2\int_{0}^{1}\int_{-\sqrt{4-4x^2}}^{\sqrt{4-4x^2}}(z+2)\,dz\,dx$$
$$=2\int_{0}^{1}\left[\frac{1}{2}z^2+2z\right]_{z=-2\sqrt{1-x^2}}^{z=2\sqrt{1-x^2}}dx$$
$$=2\int_0^1 8\sqrt{1-x^2}\,dx$$
$$=16\left[\tfrac{1}{2}x\sqrt{1-x^2}+\tfrac{1}{2}\sin^{-1}x\right]_0^1=4\pi$$

14.



$$V=\int_0^1\int_{-\sqrt{x}}^{\sqrt{x}}\int_0^{1-x}dz\,dy\,dx=\int_0^1\int_{-\sqrt{x}}^{\sqrt{x}}(1-x)\,dy\,dx$$
$$=\int_0^1 2\sqrt{x}\,(1-x)\,dx=\int_0^1 2\left(\sqrt{x}-x^{3/2}\right)dx$$
$$=2\left[\tfrac{2}{3}x^{3/2}-\tfrac{2}{5}x^{5/2}\right]_0^1=2\left(\tfrac{2}{3}-\tfrac{2}{5}\right)=\tfrac{8}{15}$$

15. The paraboloids $z=x^2+y^2$ and $z=18-x^2-y^2$ intersect when $x^2+y^2=18-x^2-y^2 \Rightarrow 2x^2+2y^2=18 \Rightarrow x^2+y^2=9$. Thus,

$$E=\left\{(x,y,z)\mid x^2+y^2\le 9,\; x^2+y^2\le z\le 18-x^2-y^2\right\}$$

Let $D=\left\{(x,y)\mid x^2+y^2\le 9\right\}$. Then

$$V=\iiint_E dV=\iint_D\left(\int_{x^2+y^2}^{18-x^2-y^2}dz\right)dA$$
$$=\iint_D\left(18-2x^2-2y^2\right)dA$$
$$=\int_0^{2\pi}\int_0^3\left(18-2r^2\right)r\,dr\,d\theta$$
$$=\int_0^{2\pi}\left[9r^2-\tfrac{1}{2}r^4\right]_{r=0}^{r=3}d\theta=\int_0^{2\pi}\tfrac{81}{2}\,d\theta=81\pi$$

16. A CAS gives

$$\int_0^2\int_{-1}^{\sin x}\int_{z-x}^{z+x}e^{3x}\,(5y+2z)\;dy\,dz\,dx$$
$$=e^6\left(-\tfrac{35}{18}-\tfrac{511}{338}\cos 4-\tfrac{140}{169}\sin 4\right)-\tfrac{749}{1521}.$$

In Maple, we use

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int(int(int(f,y=z-x..z+x),z=-1..sin(x)),
                                  x=0..2);.
```

17. (a) $m=\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{4y^2+4z^2}^{4}\left(x^2+y^2+z^2\right)dx\,dz\,dy$

(b) $(\overline{x},\overline{y},\overline{z})$ where

$$\overline{x}=\tfrac{1}{m}\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{4y^2+4z^2}^{4}x\left(x^2+y^2+z^2\right)dx\,dz\,dy$$
$$\overline{y}=\tfrac{1}{m}\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{4y^2+4z^2}^{4}y\left(x^2+y^2+z^2\right)dx\,dz\,dy$$
$$\overline{z}=\tfrac{1}{m}\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{4y^2+4z^2}^{4}z\left(x^2+y^2+z^2\right)dx\,dz\,dy$$

(c)

$$I_z=\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{4y^2+4z^2}^{4}\left(x^2+y^2\right)\left(x^2+y^2+z^2\right)dx\,dz\,dy$$

18. $V(E)=\frac{(1)(1)(1)}{6}=\frac{1}{6}$. The equation of the plane through the last three vertices is $x+y+z=1$, so

$$f_{\text{ave}}=\tfrac{1}{1/6}\int_0^1\int_0^{1-x}\int_0^{1-x-y}(x+y+z)\,dz\,dy\,dx$$
$$=6\int_0^1\int_0^{1-x}\left[(x+y)(1-x-y)+\tfrac{1}{2}(1-x-y)^2\right]dy\,dx$$
$$=3\int_0^1\int_0^{1-x}\left(1-2xy-x^2-y^2\right)dy\,dx$$
$$=3\int_0^1\int_0^{1-x}\left[1-(x+y)^2\right]dy\,dx$$
$$=3\int_0^1\left[y-\tfrac{1}{3}(x+y)^3\right]_{y=0}^{y=1-x}dx$$
$$=3\int_0^1\left(1-x-\tfrac{1}{3}+\tfrac{1}{3}x^3\right)dx=\int_0^1\left(x^3-3x+2\right)dx$$
$$=\tfrac{1}{4}-\tfrac{3}{2}+2=\tfrac{3}{4}$$