12.6 Surface Area

A Click here for answers.

S Click here for solutions.

1–3 Find the area of the surface.

1. The part of the plane $x + 2y + z = 4$ that lies inside the cylinder $x^2 + y^2 = 4$

2. The part of the plane $2x + 3y - z + 1 = 0$ that lies above the rectangle $[1, 4] \times [2, 4]$

3. The part of the surface $z = x + y^2$ that lies above the triangle with vertices $(0, 0)$, $(1, 1)$, and $(0, 1)$

4. Set up, but do not evaluate, an integral for the area of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

Answers

E Click here for exercises.

S Click here for solutions.

1. $4\sqrt{6}\,\pi$

2. $6\sqrt{14}$

3. $\frac{3}{\sqrt{6}} - \frac{1}{3\sqrt{2}}$

4. $A(S) = 2\int_{-a}^{a}\int_{-(b/a)\sqrt{a^2-x^2}}^{(b/a)\sqrt{a^2-x^2}} |\mathbf{r}_x \times \mathbf{r}_y|\,dy\,dx$ where

$$|\mathbf{r}_x \times \mathbf{r}_y| = \frac{1}{ab}\sqrt{\frac{a^2b^2\left(a^2b^2 - b^2x^2 - a^2y^2\right) + c^2b^4x^2 + c^2a^4y^2}{a^2b^2 - b^2x^2 - a^2y^2}}$$

Solutions

E Click here for exercises.

A Click here for answers.

1. Here $z = f(x,y) = 4 - x - 2y$ with $0 \le x^2 + y^2 \le 4$. Thus by (2)

$$A(S) = \iint_D \sqrt{(-1)^2 + (-2)^2 + 1}\, dA$$
$$= \sqrt{6} \iint_{x^2+y^2 \le 4} dA = \sqrt{6}\pi (2)^2 = 4\sqrt{6}\,\pi$$

2. $z = f(x,y) = 1 + 2x + 3y$ and so, by (2),

$$A(S) = \int_1^4 \int_2^4 \sqrt{2^2 + 3^2 + 1}\, dy\, dx$$
$$= \sqrt{14}\,(4-1)(4-2) = 6\sqrt{14}$$

3. $z = f(x,y) = x + y^2$ with $0 \le x \le y$, $0 \le y \le 1$. Thus, by Formula 3,

$$A(S) = \iint_D \sqrt{1 + 1 + 4y^2}\, dA = \int_0^1 \int_0^y \sqrt{2 + 4y^2}\, dx\, dy$$
$$= \int_0^1 \left[x\sqrt{2+4y^2}\right]_{x=0}^{x=y} dy = \int_0^1 y\sqrt{2+4y^2}\, dy$$
$$= 2\left(\tfrac{1}{24}\right)\left(2+4y^2\right)^{3/2}\Big]_0^1 = \tfrac{1}{12}\left(6\sqrt{6} - 2\sqrt{2}\right)$$
$$= \tfrac{3}{\sqrt{6}} - \tfrac{1}{3\sqrt{2}}$$

4. Let S_1 be that portion of S which lies above the plane $z = 0$. Then $A(S) = 2A(S_1)$ and S_1 is given by

$z = \dfrac{c}{ab}\sqrt{a^2b^2 - b^2x^2 - a^2y^2}$, where $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \le 1$ or $b^2x^2 + a^2y^2 \le a^2b^2$. Now

$$|\mathbf{r}_x \times \mathbf{r}_y| = \left[1 + \frac{c^2b^2x^2}{a^2}\left(a^2b^2 - b^2x^2 - a^2y^2\right) + \frac{c^2a^2y^2}{b^2}\left(a^2b^2 - b^2x^2 - a^2y^2\right)\right]^{1/2}$$
$$= \frac{1}{ab}\sqrt{\frac{a^2b^2\left(a^2b^2 - b^2x^2 - a^2y^2\right) + c^2b^4x^2 + c^2a^4y^2}{a^2b^2 - b^2x^2 - a^2y^2}}$$

Then $A(S) = 2\int_{-a}^{a} \int_{-(b/a)\sqrt{a^2-x^2}}^{(b/a)\sqrt{a^2-x^2}} |\mathbf{r}_x \times \mathbf{r}_y|\, dy\, dx$ with $|\mathbf{r}_x \times \mathbf{r}_y|$ given above.

Alternate Solution: Let S_1 be as above. Then in spherical coordinates, $x = a\sin\phi\cos\theta$, $y = b\sin\phi\sin\theta$, and $z = c\cos\phi$, where $0 \le \phi \le \frac{\pi}{2}$ and $0 \le \theta \le 2\pi$. Then following Example 10,

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = bc\sin^2\phi\cos\theta\,\mathbf{i} + ac\sin^2\phi\sin\theta\,\mathbf{j} + ab\sin\phi\cos\phi\,\mathbf{k}$$

Hence,

$$A(S) = 2A(S_1)$$
$$= 2\int_0^{2\pi}\int_0^{\pi/2} \sqrt{b^2c^2\sin^4\phi\cos^2\theta + a^2c^2\sin^4\phi\sin^2\theta + a^2b^2\sin^2\phi\cos^2\phi}\; d\phi\, d\theta$$