# 12.2 Iterated Integrals

A Click here for answers.

S Click here for solutions.

**1–4** Find $\int_0^2 f(x, y)\,dy$ and $\int_0^1 f(x, y)\,dx$.

**1.** $f(x, y) = x^2y^3$

**2.** $f(x, y) = 2xy - 3x^2$

**3.** $f(x, y) = xe^{x+y}$

**4.** $f(x, y) = \dfrac{x}{y^2 + 1}$

**5–12** Calculate the iterated integral.

**5.** $\int_0^4 \int_0^2 x\sqrt{y}\,dx\,dy$

**6.** $\int_0^2 \int_0^3 e^{x-y}\,dy\,dx$

**7.** $\int_{-1}^1 \int_0^1 (x^3y^3 + 3xy^2)\,dy\,dx$

**8.** $\int_0^1 \int_1^2 (x^4 - y^2)\,dx\,dy$

**9.** $\int_0^{\pi/4} \int_0^3 \sin x\,dy\,dx$

**10.** $\int_0^{\pi/2} \int_0^{\pi/2} \sin x\,\cos y\,dy\,dx$

**11.** $\int_0^3 \int_0^1 \sqrt{x + y}\,dx\,dy$

**12.** $\int_0^{\pi/2} \int_0^{\pi/2} \sin(x + y)\,dy\,dx$

**13–19** Calculate the double integral.

**13.** $\iint_R (2y^2 - 3xy^3)\,dA, \quad R = \{(x, y) \mid 1 \leqslant x \leqslant 2,\ 0 \leqslant y \leqslant 3\}$

**14.** $\iint_R \left(xy^2 + \dfrac{y}{x}\right) dA, \quad R = \{(x, y) \mid 2 \leqslant x \leqslant 3,\ -1 \leqslant y \leqslant 0\}$

**15.** $\iint_R x \sin y\,dA, \quad R = \{(x, y) \mid 1 \leqslant x \leqslant 4,\ 0 \leqslant y \leqslant \pi/6\}$

**16.** $\iint_R \dfrac{1 + x}{1 + y}\,dA, \quad R = \{(x, y) \mid -1 \leqslant x \leqslant 2,\ 0 \leqslant y \leqslant 1\}$

**17.** $\iint_R xye^y\,dA, \quad R = \{(x, y) \mid 0 \leqslant x \leqslant 2,\ 0 \leqslant y \leqslant 1\}$

**18.** $\iint_R xe^{xy}\,dA, \quad R = [0, 1] \times [0, 1]$

**19.** $\iint_R \dfrac{1}{x + y}\,dA, \quad R = [1, 2] \times [0, 1]$

**20.** Find the volume of the solid lying under the plane $z = 2x + 5y + 1$ and above the rectangle $\{(x, y) \mid -1 \leqslant x \leqslant 0,\ 1 \leqslant y \leqslant 4\}$.

**21.** Find the volume of the solid lying under the circular paraboloid $z = x^2 + y^2$ and above the rectangle $R = [-2, 2] \times [-3, 3]$.

**22.** Find the volume of the solid lying under the hyperbolic paraboloid $z = y^2 - x^2$ and above the square $R = [-1, 1] \times [1, 3]$.

**23.** Find the average value of $f(x, y) = x \sin xy$ over the rectangle $R = [0, \pi/2] \times [0, 1]$.

## Answers

E Click here for exercises.

S Click here for solutions.

**1.** $4x^2, \frac{1}{3}y^3$

**2.** $4x - 6x^2, y - 1$

**3.** $xe^x\left(e^2 - 1\right), e^y$

**4.** $x\tan^{-1}2, \dfrac{1}{2\left(y^2+1\right)}$

**5.** $\frac{32}{3}$

**6.** $e^2 - e^{-1} - 1 + e^{-3}$

**7.** $0$

**8.** $\frac{88}{15}$

**9.** $3\left(1 - \frac{1}{\sqrt{2}}\right)$

**10.** $1$

**11.** $\frac{4}{15}\left(31 - 9\sqrt{3}\right)$

**12.** $2$

**13.** $-\frac{585}{8}$

**14.** $\frac{5}{6} + \ln\sqrt{\frac{2}{3}}$

**15.** $\frac{15\left(2-\sqrt{3}\right)}{4}$

**16.** $\frac{9}{2}\ln 2$

**17.** $2$

**18.** $e - 2$

**19.** $\ln\frac{27}{16}$

**20.** $\frac{75}{2}$

**21.** $104$

**22.** $16$

**23.** $1 - \frac{2}{\pi}$

## Solutions

E Click here for exercises.

A Click here for answers.

**1.** $\int_0^2 x^2y^3\,dy = x^2\left[\frac{1}{4}y^4\right]_0^2 = 4x^2$,

$\int_0^1 x^2y^3\,dx = y^3\left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{3}y^3$

**2.** $\int_0^2 \left(2xy - 3x^2\right)dy = \left[xy^2 - 3x^2y\right]_0^2 = 4x - 6x^2$,

$\int_0^1 \left(2xy - 3x^2\right)dx = \left[yx^2 - x^3\right]_0^1 = y - 1$

**3.** $\int_0^2 xe^{x+y}\,dy = xe^x\left[e^y\right]_0^2 = x\left(e^{x+2} - e^x\right)$

$= xe^x\left(e^2 - 1\right)$

$\int_0^1 xe^{x+y}\,dx = e^y\int_0^1 xe^x\,dx = e^y\left[xe^x - e^x\right]_0^1 = e^y$

**4.** $\displaystyle\int_0^2 \frac{x}{y^2+1}\,dy = x\left[\tan^{-1}y\right]_0^2 = x\tan^{-1}2$,

$\displaystyle\int_0^1 \frac{x}{y^2+1}\,dx = \frac{1}{y^2+1}\left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2\left(y^2+1\right)}$

**5.** $\int_0^4\int_0^2 x\sqrt{y}\,dx\,dy = \int_0^4 \sqrt{y}\left[\frac{1}{2}x^2\right]_0^2 dy = \int_0^4 2\sqrt{y}\,dy$

$= \left[\frac{4}{3}y^{3/2}\right]_0^4 = \frac{32}{3}$

**6.** $\int_0^2\int_0^3 e^{x-y}\,dy\,dx = \int_0^2\left[-e^{x-y}\right]_0^3 dx$

$= \int_0^2 e^x\left(1 - e^{-3}\right)dx = e^2 - e^{-1} - 1 + e^{-3}$

**7.** $\int_{-1}^1\int_0^1\left(x^3y^2 + 3xy^2\right)dy\,dx$

$= \int_{-1}^1\left[\frac{1}{4}x^3y^4 + xy^3\right]_{y=0}^{y=1} dx = \int_{-1}^1\left[\frac{1}{4}x^3 + x\right]dx$

$= \left[\frac{1}{16}x^4 + \frac{1}{2}x^2\right]_{-1}^1 = 0$

*Alternate Solution:* Applying Fubini's Theorem, the integral equals

$\int_0^1\int_{-1}^1\left(x^3y^2 + 3xy^2\right)dx\,dy$

$= \int_0^1\left[\frac{1}{4}y^2x^4 + \frac{3}{2}y^2x^2\right]_{x=-1}^{x=1} dy = \int_0^1 0\,dy = 0$

**8.** $\int_0^1\int_1^2\left(x^4 - y^2\right)dx\,dy = \int_1^2\int_0^1\left(x^4 - y^2\right)dy\,dx$

$= \int_1^2\left[x^4y - \frac{1}{3}y^3\right]_{y=0}^{y=1} dx = \int_1^2\left[x^4 - \frac{1}{3}\right]dx$

$= \left[\frac{1}{5}x^5 - \frac{1}{3}x\right]_1^2 = \frac{88}{15}$

**9.** $\int_0^{\pi/4}\int_0^3 \sin x\,dy\,dx = 3\int_0^{\pi/4}\sin x\,dx = 3\left[-\cos x\right]_0^{\pi/4}$

$= 3\left(1 - \frac{1}{\sqrt{2}}\right)$

**10.** $\int_0^{\pi/2}\int_0^{\pi/2}\sin x\cos y\,dy\,dx$

$= \int_0^{\pi/2}\sin x\,dx\int_0^{\pi/2}\cos y\,dy$ (as in Example 5)

$= \left[-\cos x\right]_0^{\pi/2}\left[\sin y\right]_0^{\pi/2} = -(0-1)(1-0) = 1$

**11.** $\int_0^3\int_0^1\sqrt{x+y}\,dx\,dy = \int_0^3\left[\frac{2}{3}(x+y)^{3/2}\right]_{x=0}^{x=1} dy$

$= \frac{2}{3}\int_0^3\left[(1+y)^{3/2} - y^{3/2}\right]dy$

$= \frac{2}{3}\left[\frac{2}{5}(1+y)^{5/2} - \frac{2}{5}y^{5/2}\right]_0^3$

$= \frac{4}{15}\left[32 - 3^{5/2} - 1\right] = \frac{4}{15}\left(31 - 9\sqrt{3}\right)$

**12.** $\int_0^{\pi/2}\int_0^{\pi/2}\sin(x+y)\,dy\,dx$

$= \int_0^{\pi/2}\left[-\cos(x+y)\right]_{y=0}^{y=\pi/2} dx$

$= \int_0^{\pi/2}\left[\cos x - \cos\left(x + \frac{\pi}{2}\right)\right]dx$

$= \left[\sin x - \sin\left(x + \frac{\pi}{2}\right)\right]_0^{\pi/2}$

$= (1-0) - (0-1) = 2$

**13.** $\int_1^2\int_0^3\left(2y^2 - 3xy^3\right)dy\,dx = \int_1^2\left[\frac{2}{3}y^3 - \frac{3}{4}xy^4\right]_{y=0}^{y=3} dx$

$= \int_1^2\left(18 - \frac{243}{4}x\right)dx = \left[18x - \frac{243}{8}x^2\right]_1^2 = -\frac{585}{8}$

**14.** $\int_2^3\int_{-1}^0\left(xy^2 + yx^{-1}\right)dy\,dx$

$= \int_2^3\left[\frac{1}{3}xy^3 + \frac{1}{2}y^2x^{-1}\right]_{y=-1}^{y=0} dx = \int_2^3\left(\frac{1}{3}x - \frac{1}{2}x^{-1}\right)dx$

$= \left[\frac{1}{6}x^2 - \frac{1}{2}\ln x\right]_2^3 = \frac{5}{6} + \ln\sqrt{\frac{2}{3}}$

**15.** $\int_0^{\pi/6}\int_1^4 x\sin y\,dx\,dy = \left(\int_0^{\pi/6}\sin y\,dy\right)\left(\int_1^4 x\,dx\right)$

$= \left(1 - \frac{\sqrt{3}}{2}\right)\frac{15}{2} = \frac{15\left(2-\sqrt{3}\right)}{4}$

**16.** $\displaystyle\left[\int_0^1 \frac{1}{1+y}\,dy\right]\left[\int_{-1}^2 (1+x)\,dx\right]$

$= \left[\ln(1+y)\right]_0^1\left[x + \frac{1}{2}x^2\right]_{-1}^2$

$= (\ln 2)\left(2 + 2 + 1 - \frac{1}{2}\right) = \frac{9}{2}\ln 2$

**17.** $\iint_R xye^y\,dA = \int_0^2\int_0^1 xye^y\,dy\,dx = \int_0^2 x\,dx\int_0^1 ye^y\,dy$

$= \left[\frac{1}{2}x^2\right]_0^2\left[e^y(y-1)\right]_0^1$ (by parts)

$= \frac{1}{2}(4-0)\left(0 + e^0\right) = 2$

**18.** $\int_0^1\int_0^1 xe^{xy}\,dy\,dx = \int_0^1\left[e^{xy}\right]_{y=0}^{y=1} dx = \int_0^1\left(e^x - 1\right)dx$

$= \left[e^x - x\right]_0^1 = e - 2$

**19.** $\displaystyle\int_0^1\int_1^2 \frac{1}{x+y}\,dx\,dy = \int_0^1\left[\ln(x+y)\right]_{x=1}^{x=2} dy$

$= \int_0^1\left[\ln(2+y) - \ln(1+y)\right]dy$

$= \Big[\left[(2+y)\ln(2+y) - (2+y)\right]$

$- \left[(1+y)\ln(1+y) - (1+y)\right]\Big]_0^1$

[by parts separately for each term or by the Table of Integrals]

$= (3\ln 3) - 3 - (2\ln 2) + 2 - \left[(2\ln 2 - 2) - (0 - 1)\right]$

$= 3\ln 3 - 4\ln 2 = \ln\frac{27}{16}$

**20.** $V = \iint_R (2x + 5y + 1)\,dA = \int_1^4\int_{-1}^0 (2x + 5y + 1)\,dx\,dy$

$= \int_1^4\left[x^2 + 5xy + x\right]_{x=-1}^{x=0} dy = \int_1^4 5y\,dy = \frac{5}{2}y^2\Big]_1^4$

$= \frac{75}{2}$

**21.** $V = \iint_R \left(x^2 + y^2\right) dA = \int_{-3}^{3}\int_{-2}^{2} \left(x^2 + y^2\right) dx\, dy$

$= \int_{-3}^{3} \left[\frac{1}{3}x^3 + y^2 x\right]_{x=-2}^{x=2} dy = \int_{-3}^{3} \left[\frac{16}{3} + 4y^2\right] dy$

$= \left[\frac{16}{3}y + \frac{4}{3}y^3\right]_{-3}^{3} = 2\,(16 + 36) = 104$

**22.** $V = \int_1^3 \int_{-1}^{1} \left(y^2 - x^2\right) dx\, dy = 2\int_1^3 \int_0^1 \left(y^2 - x^2\right) dx\, dy$

$= 2\int_1^3 \left[y^2 x - \frac{1}{3}x^3\right]_{x=0}^{x=1} dy = 2\int_1^3 \left(y^2 - \frac{1}{3}\right) dy$

$= \frac{2}{3}\left[y^3 - y\right]_1^3 = 16$

**23.** $A\,(R) = \frac{\pi}{2} \cdot 1 = \frac{\pi}{2}$, so

$$f_{\text{ave}} = \frac{1}{A\,(R)} \iint_R f\,(x, y)\, dA$$

$$= \frac{1}{\pi/2} \int_0^{\pi/2} \int_0^1 x \sin xy\, dy\, dx$$

$$= \frac{2}{\pi} \int_0^{\pi/2} \left[-\cos xy\right]_{y=0}^{y=1} dx$$

$$= \frac{2}{\pi} \int_0^{\pi/2} (1 - \cos x)\, dx$$

$$= \frac{2}{\pi} \left[x - \sin x\right]_0^{\pi/2} = 1 - \frac{2}{\pi}$$