11.8 Lagrange Multipliers

A Click here for answers.

S Click here for solutions.

1–4 Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint(s).

1. $f(x, y) = 2x + y$; $x^2 + 4y^2 = 1$

2. $f(x, y) = xy$; $9x^2 + y^2 = 4$

3. $f(x, y, z) = x + 3y + 5z$;
$x^2 + y^2 + z^2 = 1$

4. $f(x, y, z) = x - y + 3z$;
$x^2 + y^2 + 4z^2 = 4$

5–8 Use Lagrange multipliers to solve the problem.

5. Find the shortest distance from the point $(2, -2, 3)$ to the plane $6x + 4y - 3z = 2$.

6. Find the point on the plane $2x - y + z = 1$ that is closest to the point $(-4, 1, 3)$.

7. Find the point on the plane $x + 2y + 3z = 4$ that is closest to the origin.

8. Find the shortest distance from the point (x_0, y_0, z_0) to the plane $Ax + By + Cz + D = 0$.

Answers

E Click here for exercises.

S Click here for solutions.

1. Maximum $f\left(\frac{4}{\sqrt{17}}, \frac{1}{2\sqrt{17}}\right) = \frac{\sqrt{17}}{2}$,
   minimum $f\left(-\frac{4}{\sqrt{17}}, -\frac{1}{2\sqrt{17}}\right) = -\frac{\sqrt{17}}{2}$

2. Maximum $f\left(\frac{\sqrt{2}}{3}, \sqrt{2}\right) = f\left(-\frac{\sqrt{2}}{3}, -\sqrt{2}\right) = \frac{2}{3}$,
   minimum $f\left(\frac{\sqrt{2}}{3}, -\sqrt{2}\right) = f\left(-\frac{\sqrt{2}}{3}, \sqrt{2}\right) = -\frac{2}{3}$

3. Maximum $f\left(\frac{1}{\sqrt{35}}, \frac{3}{\sqrt{35}}, \frac{5}{\sqrt{35}}\right) = \sqrt{35}$,
   minimum $f\left(-\frac{1}{\sqrt{35}}, -\frac{3}{\sqrt{35}}, -\frac{5}{\sqrt{35}}\right) = -\sqrt{35}$

4. Maximum $f\left(\frac{4}{\sqrt{17}}, -\frac{4}{\sqrt{17}}, \frac{3}{\sqrt{17}}\right) = \sqrt{17}$,
   minimum $f\left(-\frac{4}{\sqrt{17}}, \frac{4}{\sqrt{17}}, -\frac{3}{\sqrt{17}}\right)$

5. $\frac{7}{\sqrt{61}}$

6. $\left(-\frac{5}{3}, -\frac{1}{6}, \frac{25}{6}\right)$

7. $\left(\frac{2}{7}, \frac{4}{7}, \frac{6}{7}\right)$

8. $\dfrac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$

Solutions

E Click here for exercises.

A Click here for answers.

1. $f(x,y) = 2x + y$, $g(x,y) = x^2 + 4y^2 = 1 \Rightarrow \nabla f = \langle 2, 1\rangle$, $\lambda \nabla g = \langle 2\lambda x, 8\lambda y\rangle$. Then $\lambda x = 1$ and $8\lambda y = 1$ imply $x = \dfrac{1}{\lambda}$, $y = \dfrac{1}{8\lambda}$. But $1 = x^2 + 4y^2 = \dfrac{1}{\lambda^2} + 4\left(\dfrac{1}{64\lambda^2}\right)$ or $\lambda^2 = \frac{17}{16}$, so $\lambda = \pm\frac{\sqrt{17}}{4}$. Thus the possible points are $\left(\pm\frac{4}{\sqrt{17}}, \frac{1}{2\sqrt{17}}\right)$, $\left(\pm\frac{4}{\sqrt{17}}, -\frac{1}{2\sqrt{17}}\right)$. Since f is linear in x and y the maximum value of f on the ellipse is $f\left(\frac{4}{\sqrt{17}}, \frac{1}{2\sqrt{17}}\right) = \frac{\sqrt{17}}{2}$ and the minimum value is $f\left(-\frac{4}{\sqrt{17}}, -\frac{1}{2\sqrt{17}}\right) = -\frac{\sqrt{17}}{2}$.

2. $f(x,y) = xy$, $g(x,y) = 9x^2 + y^2 = 4 \Rightarrow \nabla f = \langle y, x\rangle$, $\lambda\nabla g = \langle 18\lambda x, 2\lambda y\rangle$. Then $y = 18\lambda x$ implies $(x,y) = (0,0)$ or $\lambda = y/18x$ and $x = 2\lambda y$ implies $(x,y) = (0,0)$ or $\lambda = \dfrac{x}{2y}$. Thus $(x,y) = (0,0)$ or $\dfrac{y}{18x} = \dfrac{x}{2y}$ implies $y^2 = 9x^2$. Now $(x,y) = (0,0)$ doesn't satisfy $g(x,y) = 4$, and when $y^2 = 9x^2$, $g(x,y) = 4$ implies $x^2 = \frac{2}{9}$ or $x = \pm\frac{\sqrt{2}}{3}$. Hence the possible points are $\left(\pm\frac{\sqrt{2}}{3}, \sqrt{2}\right)$, $\left(\pm\frac{\sqrt{2}}{3}, -\sqrt{2}\right)$ and the maximum value of f on the ellipse is $f\left(\frac{\sqrt{2}}{3}, \sqrt{2}\right) = f\left(-\frac{\sqrt{2}}{3}, -\sqrt{2}\right) = \frac{2}{3}$ while the minimum value is $f\left(-\frac{\sqrt{2}}{3}, \sqrt{2}\right) = f\left(\frac{\sqrt{2}}{3}, -\sqrt{2}\right) = -\frac{2}{3}$.

3. $f(x,y,z) = x + 3y + 5z$, $g(x,y,z) = x^2 + y^2 + z^2 = 1 \Rightarrow \nabla f = \langle 1, 3, 5\rangle$, $\lambda\nabla g = \langle 2\lambda x, 2\lambda y, 2\lambda z\rangle$. Then $\nabla f = \lambda\nabla g$ implies $\lambda = \dfrac{1}{2x} = \dfrac{3}{2y} = \dfrac{5}{2z}$ so $x = \frac{1}{5}z$, $y = \frac{3}{5}z$. Then $x^2 + y^2 + z^2 = 1$ implies $\frac{1}{25}z^2 + \frac{9}{25}z^2 + z^2 = 1$ or $z = \pm\sqrt{\frac{5}{7}}$. Thus the possible points are $\left(\pm\frac{1}{\sqrt{35}}, \pm\frac{3}{\sqrt{35}}, \pm\frac{5}{\sqrt{35}}\right)$ with the maximum being $f\left(\frac{1}{\sqrt{35}}, \frac{3}{\sqrt{35}}, \frac{5}{\sqrt{35}}\right) = \sqrt{35}$ and the minimum being $f\left(-\frac{1}{\sqrt{35}}, -\frac{3}{\sqrt{35}}, -\frac{5}{\sqrt{35}}\right) = -\sqrt{35}$.

4. $f(x,y,z) = x - y + 3z$, $g(x,y,z) = x^2 + y^2 + 4z^2 = 4 \Rightarrow \nabla f = \langle 1, -1, 3\rangle$, $\lambda\nabla g = \langle 2\lambda x, 2\lambda y, 8\lambda z\rangle$. Then $\nabla f = \lambda\nabla g$ implies $\lambda = \dfrac{1}{2x} = -\dfrac{1}{2y} = \dfrac{3}{8z}$ so $x = \frac{4}{3}z$, $y = -\frac{4}{3}z$. Then $x^2 + y^2 + 4z^2 = 4$ implies $\frac{68}{9}z^2 = 4$ or $z = \pm\frac{3}{\sqrt{17}}$. Then the maximum of f on $x^2 + y^2 + 4z^2 = 4$ is $f\left(\frac{4}{\sqrt{17}}, -\frac{4}{\sqrt{17}}, \frac{3}{\sqrt{17}}\right) = \sqrt{17}$ and the minimum is $f\left(-\frac{4}{\sqrt{17}}, \frac{4}{\sqrt{17}}, -\frac{3}{\sqrt{17}}\right) = -\sqrt{17}$.

5. $f(x,y,z) = (x-2)^2 + (y+2)^2 + (z-3)^2$, $g(x,y,z) = 6x + 4y - 3z = 2 \Rightarrow$
$$\nabla f = \langle 2(x-2), 2(y+2), 2(z-3)\rangle = \lambda\nabla g = \langle 6\lambda, 4\lambda, -3\lambda\rangle$$
so $x = 3\lambda + 2$, $y = 2\lambda - 2$, $z = -\frac{3}{2}\lambda + 3$ and $(18\lambda + 12) + (8\lambda - 8) + \frac{9}{2}\lambda - 9 = 2$ implies $\lambda = \frac{14}{61}$. Thus the shortest distance is $\sqrt{\left(\frac{42}{61}\right)^2 + \left(\frac{28}{61}\right)^2 + \left(-\frac{21}{61}\right)^2} = \frac{7}{\sqrt{61}}$.

6. $f(x,y,z) = (x+4)^2 + (y-1)^2 + (z-3)^2$, $g(x,y,z) = 2x - y + z = 1 \Rightarrow$
$$\nabla f = \langle 2(x+4), 2(y-1), 2(z-3)\rangle = \lambda\nabla g = \langle 2\lambda, -\lambda, \lambda\rangle$$
so $x = \lambda - 4$, $y = 1 - \frac{1}{2}\lambda$, $z = 3 + \frac{1}{2}\lambda$ and $2(\lambda - 4) - \left(1 - \frac{1}{2}\lambda\right) + \left(3 + \frac{1}{2}\lambda\right) = 1$ implies $\lambda = \frac{7}{3}$. Thus the point is $\left(-\frac{5}{3}, -\frac{1}{6}, \frac{25}{6}\right)$.

7. $f(x,y,z) = x^2 + y^2 + z^2$, $g(x,y,z) = x + 2y + 3z = 4$. Then $\nabla f = \langle 2x, 2y, 2z\rangle = \lambda\nabla g = \langle\lambda, 2\lambda, 3\lambda\rangle \Rightarrow x = \frac{1}{2}\lambda$, $y = \lambda$, $z = \frac{3}{2}\lambda$ and $\frac{1}{2}\lambda + 2\lambda + \frac{9}{2}\lambda = 4 \Rightarrow \lambda = \frac{4}{7}$. Hence the point closest to the origin is $\left(\frac{2}{7}, \frac{4}{7}, \frac{6}{7}\right)$.

8. $f(x,y,z) = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2$, $g(x,y,z) = Ax + By + Cz + D = 0 \Rightarrow$
$$\nabla f = \langle 2(x - x_0), 2(y - y_0), 2(z - z_0)\rangle = \lambda\nabla g = \langle A\lambda, B\lambda, C\lambda\rangle$$
so $x = \frac{1}{2}A\lambda + x_0$, $y = \frac{1}{2}B\lambda + y_0$, $z = \frac{1}{2}C\lambda + z_0$ and $\frac{1}{2}A^2\lambda + Ax_0 + \frac{1}{2}B^2\lambda + By_0 + \frac{1}{2}C^2\lambda + Cz_0 + D = 0$ or $\dfrac{\lambda}{2} = \dfrac{-Ax_0 - By_0 - Cz_0 - D}{A^2 + B^2 + C^2}$. Thus the square of the shortest distance is $\dfrac{\left(A^2 + B^2 + C^2\right)(Ax_0 + By_0 + Cz_0 + D)^2}{\left(A^2 + B^2 + C^2\right)^2}$, so the distance is $\dfrac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$.