11.6 Directional Derivatives and the Gradient Vector

A Click here for answers.

S Click here for solutions.

1–5 Find the directional derivative of f at the given point in the direction indicated by the angle θ.

1. $f(x, y) = x^2y^3 + 2x^4y$, $(1, -2)$, $\theta = \pi/3$
2. $f(x, y) = \sin(x + 2y)$, $(4, -2)$, $\theta = 3\pi/4$
3. $f(x, y) = xe^{-2y}$, $(5, 0)$, $\theta = \pi/2$
4. $f(x, y) = (x^2 - y)^3$, $(3, 1)$, $\theta = 3\pi/4$
5. $f(x, y) = y^x$, $(1, 2)$, $\theta = \pi/2$

6–9
(a) Find the gradient of f.
(b) Evaluate the gradient at the point P.
(c) Find the rate of change of f at P in the direction of the vector $\mathbf{u}$.

6. $f(x, y) = x^3 - 4x^2y + y^2$, $P(0, -1)$, $\mathbf{u} = \langle \frac{3}{5}, \frac{4}{5} \rangle$
7. $f(x, y) = e^x \sin y$, $P(1, \pi/4)$, $\mathbf{u} = \left\langle \frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$
8. $f(x, y, z) = xy^2z^3$, $P(1, -2, 1)$, $\mathbf{u} = \left\langle \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$
9. $f(x, y, z) = xy + yz^2 + xz^3$, $P(2, 0, 3)$, $\mathbf{u} = \langle -\frac{2}{3}, -\frac{1}{3}, \frac{2}{3} \rangle$

10–17 Find the directional derivative of the function at the given point in the direction of the vector $\mathbf{v}$.

10. $f(x, y) = x/y$, $(6, -2)$, $\mathbf{v} = \langle -1, 3 \rangle$
11. $f(x, y) = \sqrt{x - y}$, $(5, 1)$, $\mathbf{v} = \langle 12, 5 \rangle$
12. $g(x, y) = xe^{xy}$, $(-3, 0)$, $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$
13. $g(x, y) = e^x \cos y$, $(1, \pi/6)$, $\mathbf{v} = \mathbf{i} - \mathbf{j}$
14. $f(x, y, z) = \sqrt{xyz}$, $(2, 4, 2)$, $\mathbf{v} = \langle 4, 2, -4 \rangle$
15. $g(x, y, z) = xe^{yz} + xye^z$, $(-2, 1, 1)$, $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$
16. $g(x, y, z) = x \tan^{-1}(y/z)$, $(1, 2, -2)$, $\mathbf{v} = \mathbf{i} + \mathbf{j} - \mathbf{k}$
17. $g(x, y, z) = z^3 - x^2y$, $(1, 6, 2)$, $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k}$

18–23 Find the maximum rate of change of f at the given point and the direction in which it occurs.

18. $f(x, y) = \sqrt{x^2 + 2y}$, $(4, 10)$
19. $f(x, y) = \cos(3x + 2y)$, $(\pi/6, -\pi/8)$
20. $f(x, y) = xe^{-y} + 3y$, $(1, 0)$
21. $f(x, y) = \ln(x^2 + y^2)$, $(1, 2)$
22. $f(x, y, z) = x + y/z$, $(4, 3, -1)$
23. $f(x, y, z) = \dfrac{x}{y} + \dfrac{y}{z}$, $(4, 2, 1)$

24–30 Find equations of (a) the tangent plane and (b) the normal line to the given surface at the specified point.

24. $xy + yz + zx = 3$, $(1, 1, 1)$
25. $xyz = 6$, $(1, 2, 3)$
26. $x^2 + y^2 - z^2 - 2xy + 4xz = 4$, $(1, 0, 1)$
27. $x^2 - 2y^2 - 3z^2 + xyz = 4$, $(3, -2, -1)$
28. $xe^{yz} = 1$, $(1, 0, 5)$
29. $4x^2 + y^2 + z^2 = 24$, $(2, 2, 2)$
30. $x^2 - 2y^2 + z^2 = 3$, $(-1, 1, -2)$

Answers

E Click here for exercises.

1. $7\sqrt{3} - 16$

2. $\frac{\sqrt{2}}{2}$

3. -10

4. $-672\sqrt{2}$

5. 1

6. (a) $\left(3x^2 - 8xy\right)\mathbf{i} + \left(2y - 4x^2\right)\mathbf{j}$
(b) $-2\,\mathbf{j}$ (c) $-\frac{8}{5}$

7. (a) $e^x \sin y\,\mathbf{i} + e^x \cos y\,\mathbf{j}$
(b) $\frac{\sqrt{2}}{2}e\,(\mathbf{i} + \mathbf{j})$ (c) $\frac{1}{\sqrt{10}}e$

8. (a) $\langle y^2z^3, 2xyz^3, 3xy^2z^2\rangle$
(b) $\langle 4, -4, 12\rangle$ (c) $\frac{20}{\sqrt{3}}$

9. (a) $\langle y + z^3, x + z^2, 2yz + 3xz^2\rangle$
(b) $\langle 27, 11, 54\rangle$ (c) $\frac{43}{3}$

10. $-\frac{2\sqrt{10}}{5}$

11. $\frac{7}{52}$

12. $\frac{29}{\sqrt{13}}$

13. $\frac{1+\sqrt{3}}{2\sqrt{2}}e$

14. $\frac{1}{6}$

15. $-\frac{e\sqrt{14}}{7}$

16. $-\frac{\pi}{4\sqrt{3}}$

17. 8

18. $\frac{\sqrt{17}}{6}, \langle 4, 1\rangle$

19. $\sqrt{\frac{13}{2}}, \langle -3, -2\rangle$

S Click here for solutions.

20. $\sqrt{5}, \langle 1, 2\rangle$

21. $\frac{2\sqrt{5}}{5}, \langle 1, 2\rangle$

22. $\sqrt{11}, \langle 1, -1, -3\rangle$

23. $\frac{\sqrt{17}}{2}, \langle 1, 0, -4\rangle$

24. (a) $x + y + z = 3$
(b) $x = y = z$

25. (a) $6x + 3y + 2z = 18$
(b) $\frac{1}{6}(x-1) = \frac{1}{3}(y-2) = \frac{1}{2}(z-3)$

26. (a) $3x - y + z = 4$
(b) $\dfrac{x-1}{3} = -y = z - 1$

27. (a) $8x + 5y = 14$
(b) $\dfrac{x-3}{8} = \dfrac{y+2}{5}, z = -1$

28. (a) $x + 5y = 1$
(b) $x - 1 = \dfrac{y}{5}, z = 5$

29. (a) $4x + y + z = 12$
(b) $\dfrac{x-2}{4} = y - 2 = z - 2$

30. (a) $x + 2y + 2z + 3 = 0$
(b) $x + 1 = \dfrac{y-1}{2} = \dfrac{z+2}{2}$

Solutions

E Click here for exercises.

A Click here for answers.

1. $f(x,y)=x^2y^3+2x^4y \Rightarrow f_x(x,y)=2xy^3+8x^3y$ and $f_y(x,y)=3x^2y^2+2x^4$. If $\mathbf{u}$ is a unit vector in the direction of $\theta=\frac{\pi}{3}$, then from Equation 6,

$$D_{\mathbf{u}}f(1,-2)=f_x(1,-2)\cos\tfrac{\pi}{3}+f_y(1,-2)\sin\tfrac{\pi}{3}=(-32)\left(\tfrac{1}{2}\right)+(14)\left(\tfrac{\sqrt{3}}{2}\right)=7\sqrt{3}-16$$

2. $f(x,y)=\sin(x+2y) \Rightarrow f_x(x,y)=\cos(x+2y)$ and $f_y(x,y)=2\cos(x+2y)$. If $\mathbf{u}$ is a unit vector in the direction of $\theta=\frac{3\pi}{4}$, then from Equation 6,

$$D_{\mathbf{u}}f(4,-2)=f_x(4,-2)\cos\tfrac{3\pi}{4}+f_y(4,-2)\sin\tfrac{3\pi}{4}=(\cos 0)\left(-\tfrac{\sqrt{2}}{2}\right)+2(\cos 0)\left(\tfrac{\sqrt{2}}{2}\right)=\tfrac{\sqrt{2}}{2}$$

3. $f(x,y)=xe^{-2y} \Rightarrow f_x(x,y)=e^{-2y}$ and $f_y(x,y)=-2xe^{-2y}$. If $\mathbf{u}$ is a unit vector in the direction of $\theta=\frac{\pi}{2}$, then

$$D_{\mathbf{u}}f(5,0)=f_x(5,0)\cos\tfrac{\pi}{2}+f_y(5,0)\sin\tfrac{\pi}{2}=1\cdot 0+(-10)\,1=-10$$

4. $f(x,y)=\left(x^2-y\right)^3 \Rightarrow D_{\mathbf{u}}f(x,y)=3\left(x^2-y\right)^2(2x)\cos\frac{3\pi}{4}+3\left(x^2-y\right)^2(-1)\sin\frac{3\pi}{4}$. Thus $D_{\mathbf{u}}f(3,1)=3(8)^2(6)\left(-\frac{\sqrt{2}}{2}\right)-3(8)^2\left(\frac{\sqrt{2}}{2}\right)=-672\sqrt{2}$.

5. $f(x,y)=y^x \Rightarrow D_{\mathbf{u}}f(x,y)=\left(y^x\ln y\right)\cos\frac{\pi}{2}+\left(xy^{x-1}\right)\sin\frac{\pi}{2}=xy^{x-1}$. Thus $D_{\mathbf{u}}f(1,2)=(1)(2)^{1-1}=1$.

6. $f(x,y)=x^3-4x^2y+y^2$

(a) $\nabla f(x,y)=f_x\,\mathbf{i}+f_y\,\mathbf{j}=\left(3x^2-8xy\right)\mathbf{i}+\left(2y-4x^2\right)\mathbf{j}$

(b) $\nabla f(0,-1)=-2\,\mathbf{j}$

(c) $\nabla f(0,-1)\cdot\mathbf{u}=-\frac{8}{5}$

7. $f(x,y)=e^x\sin y$

(a) $\nabla f(x,y)=f_x\,\mathbf{i}+f_y\,\mathbf{j}=e^x\sin y\,\mathbf{i}+e^x\cos y\,\mathbf{j}$

(b) $\nabla f\left(1,\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}e(\mathbf{i}+\mathbf{j})$

(c) $\nabla f\left(1,\frac{\pi}{4}\right)\cdot\mathbf{u}=\frac{\sqrt{2}}{2}e\left(\frac{1}{\sqrt{5}}\right)=\frac{1}{\sqrt{10}}e$

8. $f(x,y,z)=xy^2z^3$

(a) $\nabla f(x,y,z)=\langle f_x(x,y,z),f_y(x,y,z),f_z(x,y,z)\rangle=\left\langle y^2z^3,2xyz^3,3xy^2z^2\right\rangle$

(b) $\nabla f(1,-2,1)=\langle 4,-4,12\rangle$

(c) $\nabla f(1,-2,1)\cdot\mathbf{u}=\frac{4}{\sqrt{3}}+\frac{4}{\sqrt{3}}+\frac{12}{\sqrt{3}}=\frac{20}{\sqrt{3}}$

9. $f(x,y,z)=xy+yz^2+xz^3$

(a) $\nabla f(x,y,z)=\langle f_x(x,y,z),f_y(x,y,z),f_z(x,y,z)\rangle=\left\langle y+z^3,x+z^2,2yz+3xz^2\right\rangle$

(b) $\nabla f(2,0,3)=\langle 27,11,54\rangle$

(c) $\nabla f(2,0,3)\cdot\mathbf{u}=\frac{1}{3}(-54-11+108)=\frac{43}{3}$

10. $f(x,y)=x/y \Rightarrow \nabla f(x,y)=\left\langle 1/y,-x/y^2\right\rangle$, $\nabla f(6,-2)=\left\langle -\frac{1}{2},-\frac{3}{2}\right\rangle$, $\mathbf{u}=\left\langle -\frac{1}{\sqrt{10}},\frac{3}{\sqrt{10}}\right\rangle$ and $D_{\mathbf{u}}f(6,-2)=\frac{1}{2\sqrt{10}}-\frac{9}{2\sqrt{10}}=-\frac{4}{\sqrt{10}}=-\frac{2\sqrt{10}}{5}$.

11. $f(x,y)=\sqrt{x-y} \Rightarrow \nabla f(x,y)=\left\langle \frac{1}{2}(x-y)^{-1/2},-\frac{1}{2}(x-y)^{-1/2}\right\rangle$, $\nabla f(5,1)=\left\langle \frac{1}{4},-\frac{1}{4}\right\rangle$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u}=\left\langle \frac{12}{13},\frac{5}{13}\right\rangle$, so $D_{\mathbf{u}}f(5,1)=\nabla f(5,1)\cdot\mathbf{u}=\frac{12}{52}-\frac{5}{52}=\frac{7}{52}$.

12. $g(x,y)=xe^{xy} \Rightarrow \nabla g(x,y)=\left\langle e^{xy}(1+xy),x^2e^{xy}\right\rangle$, $\nabla g(-3,0)=\langle 1,9\rangle$, $\mathbf{u}=\left\langle \frac{2}{\sqrt{13}},\frac{3}{\sqrt{13}}\right\rangle$ and $D_{\mathbf{u}}g(-3,0)=\frac{2}{\sqrt{13}}+\frac{27}{\sqrt{13}}=\frac{29}{\sqrt{13}}$.

13. $g(x,y)=e^x\cos y \Rightarrow \nabla g(x,y)=\langle e^x\cos y,-e^x\sin y\rangle$, $\nabla g\left(1,\frac{\pi}{6}\right)=\left\langle \frac{\sqrt{3}}{2}e,-\frac{1}{2}e\right\rangle$, $\mathbf{u}=\left\langle \frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right\rangle$ and $D_{\mathbf{u}}g\left(1,\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2\sqrt{2}}e+\frac{1}{2\sqrt{2}}e=\frac{1+\sqrt{3}}{2\sqrt{2}}e$.

14. $f(x,y,z)=\sqrt{xyz} \Rightarrow \nabla f(x,y,z)=\frac{1}{2}(xyz)^{-1/2}\langle yz,xz,xy\rangle$, $\nabla f(2,4,2)=\left\langle 1,\frac{1}{2},1\right\rangle$, $\mathbf{u}=\left\langle \frac{2}{3},\frac{1}{3},-\frac{2}{3}\right\rangle$ and $D_{\mathbf{u}}f(2,4,2)=1\cdot\frac{2}{3}+\frac{1}{2}\cdot\frac{1}{3}+1\left(-\frac{2}{3}\right)=\frac{1}{6}$.

15. $g(x,y,z)=xe^{yz}+xye^z \Rightarrow \nabla g(x,y,z)=\left\langle e^{yz}+ye^z,xze^{yz}+xe^z,xy\left(e^{yz}+e^z\right)\right\rangle$, $\nabla g(-2,1,1)=\langle 2e,-4e,-4e\rangle$, $\mathbf{u}=\frac{1}{\sqrt{14}}\langle 1,-2,3\rangle$ and

$$D_{\mathbf{u}}g(-2,1,1)=\frac{(2e)(1)}{\sqrt{14}}+\frac{(-4e)(-2)}{\sqrt{14}}+\frac{(-4e)(3)}{\sqrt{14}}=\frac{-2e}{\sqrt{14}}=-\frac{e\sqrt{14}}{7}$$

16. $g(x,y,z)=x\tan^{-1}(y/z) \Rightarrow \nabla g(x,y,z)=\left\langle \tan^{-1}(y/z),xz/\left(y^2+z^2\right),-xy/\left(y^2+z^2\right)\right\rangle$, $\nabla g(1,2,-2)=\left\langle -\frac{\pi}{4},-\frac{1}{4},-\frac{1}{4}\right\rangle$, $\mathbf{u}=\frac{1}{\sqrt{3}}\langle 1,1,-1\rangle$ and $D_{\mathbf{u}}g(1,2,-2)=\frac{(-\pi)(1)}{4\sqrt{3}}+\frac{(-1)(1)}{4\sqrt{3}}+\frac{(-1)(-1)}{4\sqrt{3}}=-\frac{\pi}{4\sqrt{3}}$

17. $g(x,y,z)=z^3-x^2y \Rightarrow \nabla g(x,y,z)=\left\langle -2xy,-x^2,3z^2\right\rangle$, $\nabla g(1,6,2)=\langle -12,-1,12\rangle$, $\mathbf{u}=\left\langle \frac{3}{13},\frac{4}{13},\frac{12}{13}\right\rangle$, and $D_{\mathbf{u}}g(1,6,2)=\frac{(-12)(3)}{13}+\frac{(-1)(4)}{13}+\frac{(12)(12)}{13}=8$.

18. $f(x,y)=\sqrt{x^2+2y} \Rightarrow \nabla f(x,y)=\left\langle \dfrac{x}{\sqrt{x^2+2y}},\dfrac{1}{\sqrt{x^2+2y}}\right\rangle$. Thus the maximum rate of change is $|\nabla f(4,10)|=\frac{\sqrt{17}}{6}$ in the direction $\left\langle \frac{2}{3},\frac{1}{6}\right\rangle$ or $\langle 4,1\rangle$.

19. $f(x,y) = \cos(3x+2y) \quad \Rightarrow$
$\nabla f(x,y) = \langle -3\sin(3x+2y), -2\sin(3x+2y)\rangle$, so the maximum rate of change is $\left|\nabla f\left(\frac{\pi}{6}, -\frac{\pi}{8}\right)\right| = \sqrt{\frac{13}{2}}$ in the direction $\left\langle -\frac{3\sqrt{2}}{2}, -\sqrt{2}\right\rangle$ or $\langle -3, -2\rangle$.

20. $f(x,y) = xe^{-y} + 3y \quad \Rightarrow \quad \nabla f(x,y) = \langle e^{-y}, 3 - xe^{-y}\rangle$,
$\nabla f(1,0) = \langle 1, 2\rangle$ is the direction of maximum rate of change and the maximum rate is $|\nabla f(1,0)| = \sqrt{5}$.

21. $f(x,y) = \ln(x^2+y^2) \quad \Rightarrow$
$\nabla f(x,y) = \left\langle \dfrac{2x}{x^2+y^2}, \dfrac{2y}{x^2+y^2}\right\rangle$, $\nabla f(1,2) = \left\langle \frac{2}{5}, \frac{4}{5}\right\rangle$.
Thus the maximum rate of change is $|\nabla f(1,2)| = \frac{2\sqrt{5}}{5}$ in the direction $\left\langle \frac{2}{5}, \frac{4}{5}\right\rangle$ or $\langle 1, 2\rangle$.

22. $f(x,y,z) = x + y/z \quad \Rightarrow \quad \nabla f(x,y,z) = \left\langle 1, \dfrac{1}{z}, -\dfrac{y}{z^2}\right\rangle$,
so the maximum rate of change is $|\nabla f(4,3,-1)| = \sqrt{11}$ in the direction $\langle 1, -1, -3\rangle$.

23. $f(x,y,z) = \dfrac{x}{y} + \dfrac{y}{z} \quad \Rightarrow$
$\nabla f(x,y,z) = \left\langle \dfrac{1}{y}, \dfrac{1}{z} - \dfrac{x}{y^2}, -\dfrac{y}{z^2}\right\rangle$, so the maximum rate of change is $|\nabla f(4,2,1)| = \frac{\sqrt{17}}{2}$ in the direction $\left\langle \frac{1}{2}, 0, -2\right\rangle$ or $\langle 1, 0, -4\rangle$.

24. $F(x,y,z) = xy + yz + zx \quad \Rightarrow$
$\nabla F(x,y,z) = \langle y+z, z+x, x+y\rangle$,
$\nabla F(1,1,1) = \langle 2,2,2\rangle$

(a) $2x + 2y + 2z = 6$ or $x + y + z = 3$

(b) $x - 1 = y - 1 = z - 1$ or $x = y = z$

25. $F(x,y,z) = xyz \quad \Rightarrow \quad \nabla F(x,y,z) = \langle yz, zx, xy\rangle$,
$\nabla F(1,2,3) = \langle 6,3,2\rangle$

(a) $6x + 3y + 2z = 18$

(b) $\frac{1}{6}(x-1) = \frac{1}{3}(y-2) = \frac{1}{2}(z-3)$

26. $F(x,y,z) = x^2 + y^2 - z^2 - 2xy + 4xz \quad \Rightarrow$
$\nabla F(x,y,z) = \langle 2x - 2y + 4z, 2y - 2x, -2z + 4x\rangle$,
$\nabla F(1,0,1) = \langle 6, -2, 2\rangle$

(a) $6(x-1) - 2(y-0) + 2(z-1) = 0$ or $3x - y + z = 4$

(b) $\dfrac{x-1}{3} = -y = z - 1$

27. $F(x,y,z) = x^2 - 2y^2 - 3z^2 + xyz \quad \Rightarrow$
$\nabla F(x,y,z) = \langle 2x + yz, -4y + xz, -6z + xy\rangle$,
$\nabla F(3,-2,-1) = \langle 8, 5, 0\rangle$

(a) $8(x-3) + 5(y+2) + 0(z+1) = 0$ or $8x + 5y = 14$

(b) $\dfrac{x-3}{8} = \dfrac{y+2}{5}$, $z = -1$

28. $F(x,y,z) = xe^{yz} \quad \Rightarrow$
$\nabla F(x,y,z) = \langle e^{yz}, xze^{yz}, xye^{yz}\rangle$,
$\nabla F(1,0,5) = \langle 1,5,0\rangle$

(a) $1(x-1) + 5(y-0) + 0(z-5) = 0$ or $x + 5y = 1$

(b) $x - 1 = \dfrac{y}{5}$, $z = 5$

29. $F(x,y,z) = 4x^2 + y^2 + z^2$, $\nabla F(2,2,2) = \langle 16, 4, 4\rangle$

(a) $16x + 4y + 4z = 48$ or $4x + y + z = 12$

(b) $\dfrac{x-2}{16} = \dfrac{y-2}{4} = \dfrac{z-2}{4}$ or $\dfrac{x-2}{4} = y - 2 = z - 2$

30. $F(x,y,z) = x^2 - 2y^2 + z^2 \quad \Rightarrow$
$\nabla F(-1,1,-2) = \langle -2, -4, -4\rangle$

(a) $-2x - 4y - 4z = 6$ or $x + 2y + 2z + 3 = 0$

(b) $x + 1 = \dfrac{y-1}{2} = \dfrac{z+2}{2}$