# 11.2 Limits and Continuity

A Click here for answers.

S Click here for solutions.

**1–22** Find the limit, if it exists, or show that the limit does not exist.

**1.** $\lim_{(x,y)\to(2,3)} (x^2y^2 - 2xy^5 + 3y)$

**2.** $\lim_{(x,y)\to(-3,4)} (x^3 + 3x^2y^2 - 5y^3 + 1)$

**3.** $\lim_{(x,y)\to(0,0)} \dfrac{x^2y^3 + x^3y^2 - 5}{2 - xy}$

**4.** $\lim_{(x,y)\to(-2,1)} \dfrac{x^2 + xy + y^2}{x^2 - y^2}$

**5.** $\lim_{(x,y)\to(\pi,\pi)} x \sin\left(\dfrac{x + y}{4}\right)$

**6.** $\lim_{(x,y)\to(1,4)} e^{\sqrt{x+2y}}$

**7.** $\lim_{(x,y)\to(0,0)} \dfrac{x - y}{x^2 + y^2}$

**8.** $\lim_{(x,y)\to(0,0)} \dfrac{2xy}{x^2 + 2y^2}$

**9.** $\lim_{(x,y)\to(0,0)} \dfrac{(x + y)^2}{x^2 + y^2}$

**10.** $\lim_{(x,y)\to(0,0)} \dfrac{8x^2y^2}{x^4 + y^4}$

**11.** $\lim_{(x,y)\to(0,0)} \dfrac{x^3 + xy^2}{x^2 + y^2}$

**12.** $\lim_{(x,y)\to(0,0)} \dfrac{xy + 1}{x^2 + y^2 + 1}$

**13.** $\lim_{(x,y)\to(0,0)} \dfrac{x^3y^2}{x^2 + y^2}$

**14.** $\lim_{(x,y)\to(2,0)} \dfrac{xy - 2y}{x^2 + y^2 - 4x + 4}$

**15.** $\lim_{(x,y)\to(0,0)} \dfrac{\sqrt{x^2y^2 + 1} - 1}{x^2 + y^2}$

**16.** $\lim_{(x,y)\to(0,1)} \dfrac{xy - x}{x^2 + y^2 - 2y + 1}$

**17.** $\lim_{(x,y)\to(1,-1)} \dfrac{x^2 + y^2 - 2x - 2y}{x^2 + y^2 - 2x + 2y + 2}$

**18.** $\lim_{(x,y,z)\to(1,2,3)} \dfrac{xz^2 - y^2z}{xyz - 1}$

**19.** $\lim_{(x,y,z)\to(2,3,0)} [xe^z + \ln(2x - y)]$

**20.** $\lim_{(x,y,z)\to(0,0,0)} \dfrac{x^2 - y^2 - z^2}{x^2 + y^2 + z^2}$

**21.** $\lim_{(x,y,z)\to(0,0,0)} \dfrac{xy + yz + zx}{x^2 + y^2 + z^2}$

**22.** $\lim_{(x,y,z)\to(0,0,0)} \dfrac{x^2y^2z^2}{x^2 + y^2 + z^2}$

**23–24** Find $h(x, y) = g(f(x, y))$ and the set on which $h$ is continuous.

**23.** $g(t) = e^{-t}\cos t,\quad f(x, y) = x^4 + x^2y^2 + y^4$

**24.** $g(z) = \sin z,\quad f(x, y) = y \ln x$

**25–38** Determine the set of points at which the function is continuous.

**25.** $F(x, y) = \dfrac{x^2 + y^2 + 1}{x^2 + y^2 - 1}$

**26.** $F(x, y) = \dfrac{x^6 + x^3y^3 + y^6}{x^3 + y^3}$

**27.** $F(x, y) = \tan(x^4 - y^4)$

**28.** $G(x, y) = e^{xy}\sin(x + y)$

**29.** $F(x, y) = \dfrac{1}{x^2 - y}$

**30.** $F(x, y) = \ln(2x + 3y)$

**31.** $G(x, y) = \sqrt{x + y} - \sqrt{x - y}$

**32.** $f(x, y, z) = \dfrac{xyz}{x^2 + y^2 - z}$

**33.** $G(x, y) = 2^{x \tan y}$

**34.** $f(x, y, z) = x \ln(yz)$

**35.** $f(x, y, z) = x + y\sqrt{x + z}$

**36.** $f(x, y) = \begin{cases} \dfrac{2x^2 - y^2}{2x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

**37.** $f(x, y) = \begin{cases} \dfrac{x^2y^3}{2x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

**38.** $f(x, y) = \begin{cases} \dfrac{xy}{x^2 + xy + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

**39.** Prove, using Definition 1, that

(a) $\lim_{(x,y)\to(a,b)} x = a$  (b) $\lim_{(x,y)\to(a,b)} y = b$

(c) $\lim_{(x,y)\to(a,b)} c = c$

**40.** Use polar coordinates to find

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2}$$

[If $(r, \theta)$ are polar coordinates of the point $(x, y)$ with $r \geqslant o$, note that $r \to 0^+$ as $(x, y) \to (0, 0)$.]

## Answers

E Click here for exercises.

S Click here for solutions.

**1.** $-927$

**2.** 86

**3.** $-\frac{5}{2}$

**4.** 1

**5.** $\pi$

**6.** $e^3$

**7.** Does not exist

**8.** Does not exist

**9.** Does not exist

**10.** Does not exist

**11.** 0

**12.** 1

**13.** 0

**14.** Does not exist

**15.** 0

**16.** Does not exist

**17.** Does not exist

**18.** $-\frac{3}{5}$

**19.** 2

**20.** Does not exist

**21.** Does not exist

**22.** 0

**23.** $h(x,y) = e^{-\left(x^4+x^2y^2+y^4\right)} \cos\left(x^4+x^2y^2+y^4\right)$, $\mathbb{R}^2$

**24.** $h(x,y) = \sin(y \ln x)$, $\{(x,y) \mid x > 0\}$

**25.** $\left\{(x,y) \mid x^2+y^2-1 \neq 0\right\}$

**26.** $\{(x,y) \mid y \neq -x\}$

**27.** $\left\{(x,y) \mid x^4 - y^4 \neq (2n+1)\frac{\pi}{2}, n \text{ an integer}\right\}$

**28.** $\mathbb{R}^2$

**29.** $\left\{(x,y) \mid y \neq x^2\right\}$

**30.** $\{(x,y) \mid 2x+3y > 0\}$

**31.** $\{(x,y) \mid |y| \leq x\}$

**32.** $\left\{(x,y,z) \mid z \neq x^2+y^2\right\}$

**33.** $\left\{(x,y) \mid y \neq (2n+1)\frac{\pi}{2}, n \text{ an integer}\right\}$

**34.** $\{(x,y,z) \mid yz > 0\}$

**35.** $\{(x,y,z) \mid x+z \geq 0\}$

**36.** $\{(x,y) \mid (x,y) \neq (0,0)\}$

**37.** $\mathbb{R}^2$

**38.** $\{(x,y) \mid (x,y) \neq (0,0)\}$

**40.** 1

# Solutions

E Click here for exercises.

A Click here for answers.

**1.** The function is a polynomial, so the limit equals $\left(2^2\right)\left(3^2\right) - 2\,(2)\left(3^5\right) + 3\,(3) = -927$.

**2.** The function is a polynomial, so the limit equals $(-3)^3 + 3\,(-3)^2\,(4)^2 - 5\,(4)^3 + 1 = 86$.

**3.** Since this is a rational function defined at $(0,0)$, the limit equals $(0+0-5)\,/\,(2-0) = -\frac{5}{2}$.

**4.** This is a rational function defined at $(-2,1)$, so the limit equals $(4-2+1)\,/\,(4-1) = 1$.

**5.** The product of two functions continuous at $(\pi,\pi)$, so the limit equals $\pi \sin\frac{\pi+\pi}{4} = \pi$.

**6.** The composition of two continuous functions, so the limit equals $e^{\sqrt{1+8}} = e^3$.

**7.** Let $f(x,y) = \dfrac{x-y}{x^2+y^2}$. First approach $(0,0)$ along the $x$-axis. Then $f(x,0) = \dfrac{x}{x^2} = \dfrac{1}{x}$ and $\lim\limits_{x\to 0} f(x,0)$ doesn't exist. Thus $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ doesn't exist.

**8.** Let $f(x,y) = \dfrac{2xy}{x^2+2y^2}$. As $(x,y)\to(0,0)$ along the $x$-axis, $f(x,y)\to 0$. But as $(x,y)\to(0,0)$ along the line $y=x$, $f(x,x) = \dfrac{2x^2}{3x^2}$, so $f(x,y)\to\frac{2}{3}$ as $(x,y)\to(0,0)$ along this line. So the limit doesn't exist.

**9.** $f(x,y) = \dfrac{(x+y)^2}{x^2+y^2}$. As $(x,y)\to(0,0)$ along the $x$-axis, $f(x,y)\to 1$. But as $(x,y)\to(0,0)$ along the line $y=x$, $f(x,x) = \dfrac{4x^2}{2x^2} = 2$ for $x\neq 0$, so $f(x,y)\to 2$. Thus, the limit does not exist.

**10.** $f(x,y) = \dfrac{8x^2y^2}{x^4+y^4}$. Approaching $(0,0)$ along the $x$-axis gives $f(x,y)\to 0$. Approaching $(0,0)$ along the line $y=x$, $f(x,x) = \dfrac{8x^4}{2x^4} = 4$ for $x\neq 0$, so along this line $f(x,y)\to 4$ as $(x,y)\to(0,0)$. Thus the limit doesn't exist.

**11.** $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3+xy^2}{x^2+y^2} = \lim\limits_{(x,y)\to(0,0)} x = 0$

**12.** Since $\dfrac{xy+1}{x^2+y^2+1}$ is a rational function defined at $(0,0)$ the limit is $\dfrac{0+1}{0+0+1} = 1$.

**13.** $f(x,y) = \dfrac{x^3y^2}{x^2+y^2}$. We use the Squeeze Theorem: $0 \le \dfrac{\left|x^3y^2\right|}{x^2+y^2} \le \left|x^3\right|$ since $y^2 \le x^2+y^2$, and $\left|x^3\right| \to 0$ as $(x,y)\to(0,0)$. So $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$.

**14.** $f(x,y) = \dfrac{xy-2y}{x^2+y^2-4x+4} = \dfrac{y\,(x-2)}{y^2+(x-2)^2}$. Then $f(x,0) = 0$ for $x\neq 2$, so $f(x,y)\to 0$ as $(x,y)\to(2,0)$ along the $x$-axis. But $f(x,x-2) = \dfrac{(x-2)(x-2)}{(x-2)^2+(x-2)^2} = \dfrac{(x-2)^2}{2\,(x-2)^2} = \dfrac{1}{2}$ for $x\neq 2$, so $f(x,y)\to\frac{1}{2}$ as $(x,y)\to(2,0)$ along the line $y = x-2$ $(x\neq 2)$. Thus, the limit doesn't exist.

**15.** We have

$$0 \le \frac{\sqrt{x^2y^2+1}-1}{x^2+y^2}$$

$$= \frac{x^2y^2}{\left(x^2+y^2\right)\left(\sqrt{x^2+y^2+1}+1\right)} \text{ (rationalize)}$$

$$\le \frac{x^2y^2}{2\left(x^2+y^2\right)} \le x^2 \text{ [since } y^2 \le 2\left(x^2+y^2\right)\text{]}$$

But $\lim\limits_{(x,y)\to(0,0)} x^2 = 0$, so, by the Squeeze Theorem, $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sqrt{x^2y^2+1}-1}{x^2+y^2} = 0$.

**16.** Let $f(x,y) = \dfrac{xy-x}{x^2+y^2-2y+1}$. Then $f(0,y) = 0$ for $y\neq 1$, so $f(x,y)\to 0$ as $(x,y)\to(0,1)$ along the $y$-axis. But $f(x,x+1) = \dfrac{x\,(x+1-1)}{x^2+(x+1-1)^2} = \dfrac{1}{2}$ for $x\neq 0$ so $f(x,y)\to\frac{1}{2}$ as $(x,y)\to(0,1)$ along the line $y=x+1$. Thus, the limit doesn't exist.

**17.** Let

$$f(x,y) = \frac{x^2+y^2-2x-2y}{x^2+y^2-2x+2y+2} = \frac{(x-1)^2+(y-1)^2-2}{(x-1)^2+(y+1)^2}$$

Then $f(1,y) = \dfrac{(y-1)^2-2}{(y+1)^2}$. Thus, as $(x,y)\to(1,-1)$ along the line $x=1$, the limit of $f(x,y)$ doesn't exist and so the limit doesn't exist.

**18.** $\lim\limits_{(x,y,z)\to(1,2,3)} \dfrac{xz^2-y^2z}{xyz-1} = \dfrac{1\cdot 3^2 - 2^2\cdot 3}{1\cdot 2\cdot 3-1} = -\dfrac{3}{5}$ since the function is continuous at $(1,2,3)$.

**19.** 
$$\lim_{(x,y,z)\to(2,3,0)} \left[xe^z + \ln(2x-y)\right] = (2)\left(e^0\right) + \ln(4-3) = 2$$
since the function is continuous at $(2,3,0)$.

**20.** Let $f(x,y,z) = \dfrac{x^2-y^2-z^2}{x^2+y^2+z^2}$. Then $f(x,0,0) = 1$ for $x\neq 0$ and $f(0,y,0) = -1$ for $y\neq 0$, so as $(x,y,z)\to(0,0,0)$ along the $x$-axis, $f(x,y,z)\to 1$ but as $(x,y,z)\to(0,0,0)$ along the $y$-axis, $f(x,y,z)\to -1$. Thus the limit doesn't exist.

**21.** Let $f(x,y,z) = \dfrac{xy+yz+zx}{x^2+y^2+z^2}$. Then $f(x,0,0) = 0$ for $x \neq 0$, so as $(x,y,z) \to (0,0,0)$ along the $x$-axis, $f(x,y,z) \to 0$. But $f(x,x,0) = x^2/\left(2x^2\right) = \frac{1}{2}$ for $x \neq 0$, so as $(x,y,z) \to (0,0,0)$ along the line $y = x, z = 0$, $f(x,y,z) \to \frac{1}{2}$. Thus the limit doesn't exist.

**22.** We can show that the limit along any line through the origin is 0 and thus suspect that this limit exists and equals 0. Let $\varepsilon > 0$ be given. We need to find $\delta > 0$ such that $\left|\dfrac{x^2y^2z^2}{x^2+y^2+z^2} - 0\right| < \varepsilon$ whenever $0 < \sqrt{x^2+y^2+z^2} < \delta$ or equivalently $\dfrac{x^2y^2z^2}{x^2+y^2+z^2} < \varepsilon$ whenever $0 < \sqrt{x^2+y^2+z^2} < \delta$. But $x^2 \leq x^2+y^2+z^2$ and similarly for $y^2$ and $z^2$, so $\dfrac{x^2y^2z^2}{x^2+y^2+z^2} \leq \dfrac{\left(x^2+y^2+z^2\right)^3}{x^2+y^2+z^2} = \left(x^2+y^2+z^2\right)^2$ for $x^2+y^2+z^2 \neq 0$. Thus choose $\delta = \varepsilon^{1/4}$ and let $0 < \sqrt{x^2+y^2+z^2} < \delta$. Then

$$\left|\frac{x^2y^2z^2}{x^2+y^2+z^2} - 0\right| \leq \left(x^2+y^2+z^2\right)^2 = \left(\sqrt{x^2+y^2+z^2}\right)^4 < \delta^4 = \left(\varepsilon^{1/4}\right)^4 = \varepsilon$$

Hence by definition $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{x^2y^2z^2}{x^2+y^2+z^2} = 0$.

*Or:* Use the Squeeze Theorem: $0 \leq \dfrac{x^2y^2z^2}{x^2+y^2+z^2} \leq x^2y^2$ since $z^2 \leq x^2+y^2+z^2$, and $x^2y^2 \to 0$ as $(x,y,z) \to (0,0,0)$.

**23.** $h(x,y) = g(f(x,y)) = g\left(x^4+x^2y^2+y^4\right)$
$= e^{-\left(x^4+x^2y^2+y^4\right)}\cos\left(x^4+x^2y^2+y^4\right)$
Since $f$ is a polynomial it is continuous throughout $\mathbb{R}^2$ and $g$ is the product of two functions, both of which are continuous on $\mathbb{R}$, $h$ is continuous on $\mathbb{R}^2$.

**24.** $h(x,y) = g(f(x,y)) = \sin(y\ln x)$. Since $f(x,y) = y\ln x$ it is continuous on its domain $\{(x,y) \mid x > 0\}$ and $g$ is continuous throughout $\mathbb{R}$. Thus $h$ is continuous on its domain $D = \{(x,y) \mid x > 0\}$, the right half-plane excluding the $y$-axis.

**25.** $F(x,y)$ is a rational function and thus is continuous on its domain $D = \left\{(x,y) \mid x^2+y^2-1 \neq 0\right\}$, that is, $F$ is continuous except on the circle $x^2+y^2 = 1$.

**26.** $F(x,y)$ is a rational function and thus is continuous on its domain $D = \left\{(x,y) \mid x^3+y^3 \neq 0\right\} = \{(x,y) \mid y \neq -x\}$, $\mathbb{R}^2$ except the line $y = -x$.

**27.** $F(x,y) = g(f(x,y))$ where $f(x,y) = x^4 - y^4$, a polynomial so continuous on $\mathbb{R}^2$ and $g(t) = \tan t$, continuous on its domain $\left\{t \mid t \neq (2n+1)\frac{\pi}{2}, n \text{ an integer}\right\}$. Thus $F$ is continuous on its domain $D = \left\{(x,y) \mid x^4 - y^4 \neq (2n+1)\frac{\pi}{2}, n \text{ an integer}\right\}$.

**28.** $G(x,y) = g(x,y)f(x,y)$ where $g(x,y) = e^{xy}$ and $f(x,y) = \sin(x+y)$ both of which are continuous on $\mathbb{R}^2$. Thus $G$ is continuous on $\mathbb{R}^2$.

**29.** $F(x,y) = \dfrac{1}{x^2-y}$ is a rational function and thus is continuous on its domain $\left\{(x,y) \mid x^2-y \neq 0\right\} = \left\{(x,y) \mid y \neq x^2\right\}$, so $F$ is continuous on $\mathbb{R}^2$ except the parabola $y = x^2$.

**30.** $F(x,y) = \ln(2x+3y) = g(f(x,y))$ where $f(x,y) = 2x+3y$, continuous on $\mathbb{R}^2$ and $g(t) = \ln t$, continuous on its domain $\{t \mid t > 0\}$. Thus $F$ is continuous on its domain $D = \{(x,y) \mid 2x+3y > 0\}$.

**31.** $G(x,y) = g_1(f_1(x,y)) - g_2(f_2(x,y))$ where $f_1(x,y) = x+y$ and $f_2(x,y) = x-y$, both of which are polynomials so continuous on $\mathbb{R}^2$, and $g_1(t) = \sqrt{t}$, $g_2(s) = \sqrt{s}$, both of which are continuous on their respective domains $\{t \mid t \geq 0\}$ and $\{s \mid s \geq 0\}$. Thus $g_1 \circ f_1$ is continuous on its domain $D_1 = \{(x,y) \mid x+y \geq 0\} = \{(x,y) \mid y \geq -x\}$ and $g_2 \circ f_2$ is continuous on its domain $D_2 = \{(x,y) \mid x-y \geq 0\} = \{(x,y) \mid y \leq x\}$. Then $G$, being the difference of these two composite functions, is continuous on its domain
$D = D_1 \cap D_2 = \{(x,y) \mid -x \leq y \leq x\}$
$= \{(x,y) \mid |y| \leq x\}$

**32.** $f(x,y,z) = \dfrac{xyz}{x^2+y^2-z}$ is a rational function and thus is continuous on its domain $\left\{(x,y,z) \mid x^2+y^2-z \neq 0\right\} = \left\{(x,y,z) \mid z \neq x^2+y^2\right\}$, so $f$ is continuous on $\mathbb{R}^3$ except on the circular paraboloid $z = x^2+y^2$.

**33.** $G(x,y) = g(f(x,y))$ where $f(x,y) = x\tan y$ which is continuous on its domain $\left\{(x,y) \mid y \neq (2n+1)\frac{\pi}{2}, n \text{ an integer}\right\}$ and $g(t) = 2^t$ which is continuous on $\mathbb{R}$. Thus $G(x,y)$ is continuous on its domain $D = \left\{(x,y) \mid y \neq (2n+1)\frac{\pi}{2}, n \text{ an integer}\right\}$.

**34.** $f(x,y,z) = xg(f(y,z))$ where $f(y,z) = yz$, continuous on $\mathbb{R}^2$ and $g(t) = \ln t$, continuous on its domain $\{t \mid t > 0\}$. Since $h(x) = x$ is continuous on $\mathbb{R}$, $f(x,y,z)$ is continuous on its domain $D = \{(x,y,z) \mid yz > 0\}$.

**35.** $f(x,y,z) = h(x) + k(y)\,g(f(x,z))$ where $h(x) = x$ and $k(y) = y$, both continuous on $\mathbb{R}$ and $f(x,z) = x + z$, continuous on $\mathbb{R}^2$, $g(t) = \sqrt{t}$ continuous on its domain $D = \{t \mid t \geq 0\}$. Thus $f$ is continuous on its domain $D = \{(x,y,z) \mid x + z \geq 0\}$.

**In Problems 36–38, each $f$ is a piecewise defined function whose first piece is a rational function defined everywhere except at the origin. Thus each $f$ is continuous on $\mathbb{R}^2$ except possibly at the origin. So for each we need only check $\lim\limits_{(x,y)\to(0,0)} f(x,y)$.**

**36.** Letting $z = \sqrt{2}x$,

$$\lim_{(x,y)\to(0,0)} \frac{2x^2 - y^2}{2x^2 + y^2} = \lim_{(z,\,y)\to(0,0)} \frac{z^2 - y^2}{z^2 + y^2}$$ which doesn't exist by Example 1. Thus $f$ is not continuous at $(0,0)$ and the largest set on which $f$ is continuous is $\{(x,y) \mid (x,y) \neq (0,0)\}$.

**37.** Since $x^2 \leq 2x^2 + y^2$, we have $\left|\dfrac{x^2y^3}{2x^2 + y^2}\right| \leq \left|y^3\right|$. We know that $\left|y^3\right| \to 0$ as $(x,y) \to (0,0)$. So, by the Squeeze Theorem, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^3}{2x^2 + y^2} = 0$. Also $f(0,0) = 0$, so $f$ is continuous at $(0,0)$. For $(x,y) \neq (0,0)$, $f(x,y)$ is equal to a rational function and is therefore continuous. Thus $f$ is continuous throughout $\mathbb{R}^2$.

**38.** Let $g(x,y) = \dfrac{xy}{x^2 + xy + y^2}$. Then $g(x,0) = 0/x^2 = 0$ for $x \neq 0$, so $g(x,y) \to 0$ as $(x,y) \to (0,0)$ along the $x$-axis. But $g(x,x) = \dfrac{x^2}{3x^2} = \frac{1}{3}$ for $x \neq 0$, so $g(x,y) \to \frac{1}{3}$ as $(x,y) \to (0,0)$ along the line $y = x$. Thus $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2 + xy + y^2}$ doesn't exist, so $f$ is not continuous at $(0,0)$ and the largest set on which $f$ is continuous is $\{(x,y) \mid (x,y) \neq (0,0)\}$.

**39.** (a) Let $\varepsilon > 0$ be given. We need to find $\delta > 0$ such that $|x - a| < \varepsilon$ whenever $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$. But $|x - a| = \sqrt{(x-a)^2} \leq \sqrt{(x-a)^2 + (y-b)^2}$. Thus setting $\delta = \varepsilon$ and letting $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$, we have $|x - a| \leq \sqrt{(x-a)^2 + (y-b)^2} < \delta = \varepsilon$. Hence, by Definition 1, $\lim\limits_{(x,y)\to(a,b)} x = a$.

(b) The argument is the same as in (a) with the roles of $x$ and $y$ interchanged.

(c) Let $\varepsilon > 0$ be given and set $\delta = \varepsilon$. Then

$$|f(x,y) - L| = |c - c| = 0 \leq \sqrt{(x-a)^2 + (y-b)^2} < \delta = \varepsilon$$

whenever $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$. Thus, by Definition 1, $\lim\limits_{(x,y)\to(a,b)} c = c$.

**40.** $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin\left(x^2 + y^2\right)}{x^2 + y^2} = \lim\limits_{r\to0^+} \dfrac{\sin\left(r^2\right)}{r^2}$, which is an indeterminate form of type $\dfrac{0}{0}$. Using l'Hospital's Rule, we get

$$\lim_{r\to0^+} \frac{\sin\left(r^2\right)}{r^2} = \lim_{r\to0^+} \frac{2r\cos\left(r^2\right)}{2r} = \lim_{r\to0^+} \cos\left(r^2\right) = 1$$

*Or:* Use the fact that $\lim\limits_{\theta\to0} \dfrac{\sin\theta}{\theta} = 1$.