10.4 Motion in Space

A Click here for answers.

S Click here for solutions.

1. Find the velocity, acceleration, and speed of a particle with position function

$$\mathbf{r}(t) = \left\langle \sqrt{t}, 1 - t \right\rangle$$

Sketch the path of the particle and draw the velocity and acceleration vectors for $t = 1$.

2–7 Find the velocity, acceleration, and speed of a particle with the given position function.

2. $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$

3. $\mathbf{r}(t) = \langle t^3, t^2 + 1, t^3 - 1 \rangle$

4. $\mathbf{r}(t) = \left\langle \sqrt{t}, t, t\sqrt{t} \right\rangle$

5. $\mathbf{r}(t) = (1/t)\,\mathbf{i} + \mathbf{j} + t^2\,\mathbf{k}$

6. $\mathbf{r}(t) = e^t\,\mathbf{i} + 2t\,\mathbf{j} + e^{-t}\,\mathbf{k}$

7. $\mathbf{r}(t) = \cosh t\,\mathbf{i} + \sinh t\,\mathbf{j} + t\,\mathbf{k}$

8. A gun has muzzle speed 120 m/s. What angle of elevation should be used to hit an object 500 m away?

9–12 Find the tangential and normal components of the acceleration vector.

9. $\mathbf{r}(t) = (t^2 + 4)\,\mathbf{i} + (2t - 3)\,\mathbf{j}$

10. $\mathbf{r}(t) = (t - \sin t)\,\mathbf{i} + (1 - \cos t)\,\mathbf{j}$

11. $\mathbf{r}(t) = t\,\mathbf{i} + 4\sin t\,\mathbf{j} + 4\cos t\,\mathbf{k}$

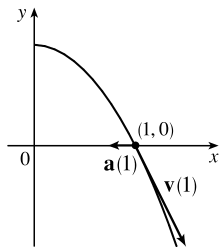
12. $\mathbf{r}(t) = t^3\,\mathbf{i} + t^2\,\mathbf{j} + t\,\mathbf{k}$

Answers

E Click here for exercises.

S Click here for solutions.

1. $\left\langle \frac{1}{2}t^{-1/2}, -1 \right\rangle$, $\left\langle -\frac{1}{4}t^{-3/2}, 0 \right\rangle$, $\sqrt{\frac{1}{4}t^{-1} + 1}$



2. $\langle 1, 2t, 3t^2 \rangle$, $\langle 0, 2, 6t \rangle$, $\sqrt{1 + 4t^2 + 9t^4}$

3. $\langle 3t^2, 2t, 3t^2 \rangle$, $\langle 6t, 2, 6t \rangle$, $|t| \sqrt{18t^2 + 4}$

4. $\left\langle \frac{1}{2}t^{-1/2}, 1, \frac{3}{2}t^{1/2} \right\rangle$, $\left\langle -\frac{1}{4}t^{-3/2}, 0, \frac{3}{4}t^{-1/2} \right\rangle$, $\frac{1}{2}\sqrt{\frac{1 + 4t + 9t^2}{t}}$

5. $\langle -t^{-2}, 0, 2t \rangle$, $\langle 2t^{-3}, 0, 2 \rangle$, $\frac{1}{t^2}\sqrt{4t^6 + 1}$

6. $\langle e^t, 2, -e^{-t} \rangle$, $\langle e^t, 0, e^{-t} \rangle$, $\sqrt{e^{2t} + 4 + e^{-2t}}$

7. $\langle \sinh t, \cosh t, 1 \rangle$, $\langle \cosh t, \sinh t, 0 \rangle$, $\sqrt{\cosh 2t + 1}$

8. 9.9°

9. $\frac{2t}{\sqrt{t^2 + 1}}$, $\frac{2}{\sqrt{t^2 + 1}}$

10. $\frac{\sin t}{\sqrt{2(1 - \cos t)}}$, $\frac{\sqrt{1 - \cos t}}{\sqrt{2}}$

11. 0, 4

12. $\frac{18t^3 + 4t}{\sqrt{9t^4 + 4t^2 + 1}}$, $\frac{2\sqrt{9t^4 + 9t^2 + 1}}{\sqrt{9t^4 + 4t^2 + 1}}$

Solutions

E Click here for exercises.

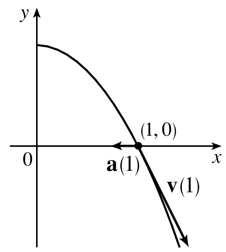
A Click here for answers.

1. $\mathbf{r}(t) = \langle \sqrt{t}, 1-t \rangle \Rightarrow \mathbf{v}(t) = \left\langle \frac{1}{2}t^{-1/2}, -1 \right\rangle$,

$\mathbf{a}(t) = \left\langle -\frac{1}{4}t^{-3/2}, 0 \right\rangle$, $|\mathbf{v}(t)| = \sqrt{\frac{1}{4}t^{-1} + 1}$

At $t = 1$: $\mathbf{v}(1) = \langle \frac{1}{2}, -1 \rangle$, $\mathbf{a}(1) = \langle -\frac{1}{4}, 0 \rangle$

Since $x^2 = t$, $y = 1 - t = 1 - x^2$, but $x = \sqrt{t}$, so $x \geq 0$.



2. $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle \Rightarrow \mathbf{v}(t) = \mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle$,

$\mathbf{a}(t) = \mathbf{v}'(t) = \langle 0, 2, 6t \rangle$,

$|\mathbf{v}(t)| = \sqrt{1^2 + (2t)^2 + (3t^2)^2} = \sqrt{1 + 4t^2 + 9t^4}$

3. $\mathbf{r}(t) = \langle t^3, t^2 + 1, t^3 - 1 \rangle \Rightarrow \mathbf{v}(t) = \langle 3t^2, 2t, 3t^2 \rangle$,

$\mathbf{a}(t) = \langle 6t, 2, 6t \rangle$,

$|\mathbf{v}(t)| = \sqrt{9t^4 + 4t^2 + 9t^4} = \sqrt{18t^4 + 4t^2}$

$= |t| \sqrt{18t^2 + 4}$

4. $\mathbf{r}(t) = \langle \sqrt{t}, t, t\sqrt{t} \rangle \Rightarrow \mathbf{v}(t) = \left\langle \frac{1}{2}t^{-1/2}, 1, \frac{3}{2}t^{1/2} \right\rangle$,

$\mathbf{a}(t) = \left\langle -\frac{1}{4}t^{-3/2}, 0, \frac{3}{4}t^{-1/2} \right\rangle$,

$|\mathbf{v}(t)| = \sqrt{\frac{1}{4}t^{-1} + 1 + \frac{9}{4}t} = \frac{1}{2}\sqrt{\frac{1 + 4t + 9t^2}{t}}$

5. $\mathbf{r}(t) = \langle 1/t, 1, t^2 \rangle \Rightarrow \mathbf{v}(t) = \langle -t^{-2}, 0, 2t \rangle$,

$\mathbf{a}(t) = \langle 2t^{-3}, 0, 2 \rangle$, $|\mathbf{v}(t)| = \sqrt{t^{-4} + 4t^2} = \frac{1}{t^2}\sqrt{4t^6 + 1}$

6. $\mathbf{r}(t) = \langle e^t, 2t, e^{-t} \rangle \Rightarrow \mathbf{v}(t) = \langle e^t, 2, -e^{-t} \rangle$,

$\mathbf{a}(t) = \langle e^t, 0, e^{-t} \rangle$, $|\mathbf{v}(t)| = \sqrt{e^{2t} + 4 + e^{-2t}}$

7. $\mathbf{r}(t) = \langle \cosh t, \sinh t, t \rangle \Rightarrow$

$\mathbf{v}(t) = \langle \sinh t, \cosh t, 1 \rangle$, $\mathbf{a}(t) = \langle \cosh t, \sinh t, 0 \rangle$,

$|\mathbf{v}(t)| = \sqrt{\sinh^2 t + \cosh^2 t + 1} = \sqrt{\cosh 2t + 1}$

Recall that $\cosh^2 t - \sinh^2 t = 1$.

8. Here the initial speed $v_0 = 120$ m/s; let α be the angle of elevation. Assuming the object is lying flat on the ground, the object will be hit at time $t = \dfrac{240 \sin \alpha}{g}$ s (again refer to Example 5). Then $\dfrac{(120)^2 \sin 2\alpha}{g} = 500$ or

$\sin 2\alpha = \dfrac{500g}{(120)^2} = \dfrac{5g}{144}$ and $2\alpha = \sin^{-1} \dfrac{5g}{144} \approx 19.9°$, so $\alpha \approx 9.9°$.

9. $\mathbf{r}'(t) = 2t\,\mathbf{i} + 2\,\mathbf{j}$, $|\mathbf{r}'(t)| = 2\sqrt{t^2 + 1}$, $\mathbf{r}''(t) = 2\,\mathbf{i}$. Thus

$a_T = \dfrac{4t}{2\sqrt{t^2+1}} = \dfrac{2t}{\sqrt{t^2+1}}$ and

$a_N = \dfrac{|-4\mathbf{k}|}{2\sqrt{t^2+1}} = \dfrac{2}{\sqrt{t^2+1}}$.

10. $\mathbf{r}'(t) = (1 - \cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j}$,

$|\mathbf{r}'(t)| = \sqrt{1 - 2\cos t + 1} = \sqrt{2(1 - \cos t)}$,

$\mathbf{r}''(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j}$. Thus $a_T = \dfrac{\sin t}{\sqrt{2(1 - \cos t)}}$ and

$a_N = \dfrac{\left|\left(\cos t - \cos^2 t - \sin^2 t\right)\mathbf{k}\right|}{\sqrt{2(1 - \cos t)}} = \dfrac{\sqrt{[(\cos t) - 1]^2}}{\sqrt{2}\sqrt{1 - \cos t}}$

$= \dfrac{1}{\sqrt{2}}\sqrt{\dfrac{(1 - \cos t)^2}{1 - \cos t}} = \dfrac{\sqrt{1 - \cos t}}{\sqrt{2}}$

11. $\mathbf{r}'(t) = \mathbf{i} + 4\cos t\,\mathbf{j} - 4\sin t\,\mathbf{k}$, $|r'(t)| = \sqrt{1 + 4} = \sqrt{5}$,

$\mathbf{r}''(t) = -4\sin t\,\mathbf{j} - 4\cos t\,\mathbf{k}$.

$a_T = -16\cos t \sin t + 16\cos t \sin t = 0$ and

$a_N = \frac{1}{\sqrt{17}} |-16\,\mathbf{i} + 4\cos t\,\mathbf{j} - 4\sin t\,\mathbf{k}|$

$= \frac{1}{\sqrt{17}}\sqrt{256 + 16} = 4$

12. $\mathbf{r}'(t) = 3t^2\,\mathbf{i} + 2t\,\mathbf{j} + \mathbf{k}$, $|\mathbf{r}'(t)| = \sqrt{9t^4 + 4t^2 + 1}$,

$\mathbf{r}''(t) = 6t\,\mathbf{i} + 2\,\mathbf{j}$. Thus $a_T = \dfrac{18t^3 + 4t}{\sqrt{9t^4 + 4t^2 + 1}}$ and

$a_N = \dfrac{\left|-2\,\mathbf{i} + 6t\,\mathbf{j} + \left(6t^2 - 12t^2\right)\mathbf{k}\right|}{\sqrt{9t^4 + 4t^2 + 1}}$

$= \dfrac{\sqrt{4 + 36t^2 + 36t^4}}{\sqrt{9t^4 + 4t^2 + 1}} = \dfrac{2\sqrt{9t^4 + 9t^2 + 1}}{\sqrt{9t^4 + 4t^2 + 1}}$