

10.3 Arc Length and Curvature

A Click here for answers.

1–3 ■ Find the length of the given curve.

1. $\mathbf{r}(t) = \langle 2t, 3 \sin t, 3 \cot t \rangle, \quad a \leq t \leq b$

2. $\mathbf{r}(t) = \langle e^t, e^t \sin t, e^t \cos t \rangle, \quad 0 \leq t \leq 2\pi$

3. $\mathbf{r}(t) = 6t \mathbf{i} + 3\sqrt{2}t^2 \mathbf{j} + 2t^3 \mathbf{k}, \quad 0 \leq t \leq 1$

4. $x = 2t, y = t^2, z = t^2, \quad 0 \leq t \leq 1$

5. $x = t \cos t, y = t \sin t, z = t, \quad 0 \leq t \leq \pi/2$

6–8 ■ Reparametrize the curve with respect to arc length measured from the point where $t = 0$ in the direction of increasing t .

6. $\mathbf{r}(t) = e^t \sin t \mathbf{i} + e^t \cos t \mathbf{j}$

7. $\mathbf{r}(t) = (1 + 2t) \mathbf{i} + (3 + t) \mathbf{j} - 5t \mathbf{k}$

8. $\mathbf{r}(t) = \cos^3 t \mathbf{i} + \sin^3 t \mathbf{j} + \cos 2t \mathbf{k}, \quad 0 \leq t \leq \pi/2$

9–14 ■

(a) Find the unit tangent and unit normal vectors $\mathbf{T}(t)$ and $\mathbf{N}(t)$.

(b) Use Formula 9 to find the curvature.

9. $\mathbf{r}(t) = \langle \sin 4t, 3t, \cos 4t \rangle$ 10. $\mathbf{r}(t) = \langle 6t, 3\sqrt{2}t^2, 2t^3 \rangle$

11. $\mathbf{r}(t) = \langle \sqrt{2} \cos t, \sin t, \sin t \rangle$

12. $\mathbf{r}(t) = \langle \frac{1}{3}t^3, t^2, 2t \rangle$

13. $\mathbf{r}(t) = \langle \sqrt{2}t, e^t, e^{-t} \rangle$ 14. $\mathbf{r}(t) = \langle t^2, 2t^3/3, t \rangle$

S Click here for solutions.

15–19 ■ Use Theorem 10 to find the curvature.

15. $\mathbf{r}(t) = \mathbf{i} + t \mathbf{j} + t^2 \mathbf{k}$

16. $\mathbf{r}(t) = (1 + t) \mathbf{i} + (1 - t) \mathbf{j} + 3t^2 \mathbf{k}$

17. $\mathbf{r}(t) = 2t^3 \mathbf{i} - 3t^2 \mathbf{j} + 6t \mathbf{k}$

18. $\mathbf{r}(t) = (t^2 + 2) \mathbf{i} + (t^2 - 4t) \mathbf{j} + 2t \mathbf{k}$

19. $\mathbf{r}(t) = \sin t \mathbf{i} + \cos t \mathbf{j} + \sin t \mathbf{k}$

20. Find the curvature of $\mathbf{r}(t) = \langle \sqrt{2}t, e^t, e^{-t} \rangle$ at the point $(0, 1, 1)$.

21–23 ■ Use Formula 11 to find the curvature.

21. $y = \sqrt{x}$

22. $y = \sin x$

23. $y = \ln x$

24–25 ■ Use the formula

$$\kappa = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{[\dot{x}^2 + \dot{y}^2]^{3/2}}$$

where the dots indicate derivatives with respect to t (see Exercise 34 in the text) to find the curvature of the parametric curve.

24. $x = t^3, \quad y = t^2$

25. $x = t \sin t, \quad y = t \cos t$

Answers

E Click here for exercises.

1. $\sqrt{13}(b-a)$
2. $\sqrt{3}(e^{2\pi}-1)$
3. 8
4. $\sqrt{3} + \frac{\sqrt{2}}{2} \ln(\sqrt{2} + \sqrt{3})$
5. $\ln\left(\sqrt{\frac{\pi^2}{4} + 2} + \frac{\pi}{2}\right) + \frac{\pi}{4} \sqrt{\frac{\pi^2}{4} + 2} - \ln \sqrt{2}$
6. $\mathbf{r}(t(s)) = \left(\frac{1}{\sqrt{2}}s + 1\right) \left[\sin\left(\ln\left(\frac{1}{\sqrt{2}}s + 1\right)\right) \mathbf{i} + \cos\left(\ln\left(\frac{1}{\sqrt{2}}s + 1\right)\right) \mathbf{j}\right]$
7. $\mathbf{r}(t(s)) = \left(1 + \frac{2}{\sqrt{30}}s\right) \mathbf{i} + \left(3 + \frac{1}{\sqrt{30}}s\right) \mathbf{j} - \frac{5}{\sqrt{30}}s \mathbf{k}$
8. $\mathbf{r}(t(s)) = \cos^3\left[\frac{1}{2}\cos^{-1}\left(1 - \frac{4}{5}s\right)\right] \mathbf{i} + \sin^3\left[\frac{1}{2}\cos^{-1}\left(1 - \frac{4}{5}s\right)\right] \mathbf{j} + \left[1 - \frac{4}{5}s\right] \mathbf{k}$
9. (a) $\frac{1}{5}\langle 4\cos 4t, 3, -4\sin 4t\rangle, \langle -\sin 4t, 0, -\cos 4t\rangle$
(b) $\frac{16}{25}$
10. (a) $\frac{1}{1+t^2}\langle 1, \sqrt{2}t, t^2\rangle, \frac{1}{\sqrt{2}(1+t^2)}\langle -2t, \sqrt{2}(1-t^2), 2t\rangle$
(b) $\frac{1}{3\sqrt{2}(1+t^2)^2}$
11. (a) $\frac{1}{\sqrt{2}}\langle -\sqrt{2}\sin t, \cos t, \cos t\rangle, \frac{1}{\sqrt{2}}\langle -\sqrt{2}\cos t, -\sin t, -\sin t\rangle$
(b) $\frac{1}{\sqrt{2}}$
12. (a) $\frac{1}{t^2+2}\langle t^2, 2t, 2\rangle, \frac{1}{t^2+2}\langle 2t, 2-t^2, -2t\rangle$
(b) $\frac{2}{(t^2+2)^2}$
13. (a) $\frac{1}{e^{2t}+1}\langle \sqrt{2}e^{2t}, e^{2t}, -1\rangle, \frac{1}{e^{2t}+1}\langle 1-e^{2t}, \sqrt{2}e^t, \sqrt{2}e^t\rangle$
(b) $\frac{\sqrt{2}e^{2t}}{(e^{2t}+1)^2}$

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14. (a) $\frac{1}{2t^2+1}\langle 2t, 2t^2, 1\rangle, \frac{1}{2t^2+1}\langle 1-2t^2, 2t, -2t\rangle$
(b) $\frac{2}{(2t^2+1)^2}$
15. $\frac{2}{(4t^2+1)^{3/2}}$
16. $\frac{3}{(1+18t^2)^{3/2}}$
17. $\frac{\sqrt{1+4t^2+t^4}}{6(t^4+t^2+1)^{3/2}}$
18. $\frac{\sqrt{6}}{2(2t^2-4t+5)^{3/2}}$
19. $\frac{\sqrt{2}}{(1+\cos^2 t)^{3/2}}$
20. $\frac{\sqrt{2}}{4}$
21. $\frac{2}{(4x+1)^{3/2}}$
22. $\frac{|\sin x|}{(1+\cos^2 x)^{3/2}}$
23. $\frac{|x|}{(x^2+1)^{3/2}}$
24. $\frac{6}{|t|(9t^2+4)^{3/2}}$
25. $\frac{2+t^2}{(1+t^2)^{3/2}}$


Solutions

E [Click here for exercises.](#)

1. $\mathbf{r}'(t) = \langle 2, 3 \cos t, -3 \sin t \rangle,$

$$|\mathbf{r}'(t)| = \sqrt{4 + 9 \cos^2 t + 9 \sin^2 t} = \sqrt{13},$$

$$L = \int_a^b \sqrt{13} dt = \sqrt{13}(b - a)$$

2. $\mathbf{r}'(t) = \langle e^t, e^t(\sin t + \cos t), e^t(\cos t - \sin t) \rangle,$

$$|\mathbf{r}'(t)| = e^t \sqrt{1 + 2 \sin^2 t + 2 \cos^2 t} = \sqrt{3}e^t,$$

$$L = \int_0^{2\pi} \sqrt{3}e^t dt = \sqrt{3}(e^{2\pi} - 1)$$

3. $\mathbf{r}'(t) = \langle 6, 6\sqrt{2}t, 6t^2 \rangle,$

$$|\mathbf{r}'(t)| = 6\sqrt{1 + 2t^2 + t^4} = 6(1 + t^2),$$

$$L = \int_0^1 6(1 + t^2) dt = \left[\frac{1}{3} \cdot 6(t + t^3) \right]_0^1 = \frac{24}{3} = 8$$

4. $\mathbf{r}'(t) = \langle 2, 2t, 2t \rangle, |\mathbf{r}'(t)| = 2\sqrt{1 + 2t^2}$

$$L = \int_0^1 \sqrt{1 + 2t^2} dt = \int_a^b \sqrt{2} \sec^3 \theta d\theta$$

$$= \frac{\sqrt{2}}{2} [\ln |\sec \theta + \tan \theta| + \tan \theta \sec \theta]_a^b$$

$$= \frac{\sqrt{2}}{2} [\ln |\sqrt{1 + 2t^2} + \sqrt{2}t| + \sqrt{2}t\sqrt{1 + 2t^2}]_0^1$$

(or use Formula 21)

$$= \frac{\sqrt{2}}{2} [\ln |\sqrt{3} + \sqrt{2}| + \sqrt{2}\sqrt{3}]$$

$$= \sqrt{3} + \frac{\sqrt{2}}{2} \ln(\sqrt{2} + \sqrt{3})$$

5. $\mathbf{r}'(t) = \langle \cos t - t \sin t, \sin t + t \cos t, 1 \rangle,$

$$|\mathbf{r}'(t)| = \sqrt{\cos^2 t + t^2 \sin^2 t + \sin^2 t + t \cos^2 t + 1} \\ = \sqrt{t^2 + 2}$$

$$L = \int_0^{\pi/2} \sqrt{t^2 + 2} dt = \int_a^b \sec^3 \theta d\theta$$

$$= [\ln |\sqrt{t^2 + 2} + t| + \frac{1}{2}t\sqrt{t^2 + 2}]_0^{\pi/2}$$

(or use Formula 21)

$$= \ln \left(\sqrt{\frac{\pi^2}{4} + 2} + \frac{\pi}{2} \right) + \frac{\pi}{4} \sqrt{\frac{\pi^2}{4} + 2} - \ln \sqrt{2}$$

6. $\mathbf{r}'(t) = e^t(\cos t + \sin t)\mathbf{i} + e^t(\cos t - \sin t)\mathbf{j},$

$$ds/dt = |\mathbf{r}'(t)| = e^t \sqrt{(\cos t + \sin t)^2 + (\cos t - \sin t)^2}$$

$$= e^t \sqrt{2 \cos^2 t + 2 \sin^2 t} = \sqrt{2}e^t$$

$$s(t) = \int_0^t |\mathbf{r}'(u)| du = \int_0^t \sqrt{2}e^u du = \sqrt{2}(e^t - 1) \Rightarrow$$

$$\frac{1}{\sqrt{2}}s + 1 = e^t \Rightarrow t(s) = \ln \left(\frac{1}{\sqrt{2}}s + 1 \right). \text{ Therefore,}$$

$$\mathbf{r}(t(s)) = \left(\frac{1}{\sqrt{2}}s + 1 \right) \left[\sin \left(\ln \left(\frac{1}{\sqrt{2}}s + 1 \right) \right) \mathbf{i} \right. \\ \left. + \cos \left(\ln \left(\frac{1}{\sqrt{2}}s + 1 \right) \right) \mathbf{j} \right]$$

7. $\mathbf{r}'(t) = 2\mathbf{i} + \mathbf{j} - 5\mathbf{k},$

$$ds/dt = |\mathbf{r}'(t)| = \sqrt{4 + 1 + 25} = \sqrt{30} \text{ and}$$

$$s(t) = \int_0^t |\mathbf{r}'(u)| du = \int_0^t \sqrt{30} du = \sqrt{30}t$$

$$\Rightarrow t(s) = \frac{1}{\sqrt{30}}s. \text{ Therefore,}$$

$$\mathbf{r}(t(s)) = \left(1 + \frac{2}{\sqrt{30}}s \right) \mathbf{i} + \left(3 + \frac{1}{\sqrt{30}}s \right) \mathbf{j} - \frac{5}{\sqrt{30}}s \mathbf{k}.$$

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8. $|\mathbf{r}'(t)|$

$$= \sqrt{(-3 \cos^2 t \sin t)^2 + (3 \sin^2 t \cos t)^2 + (-2 \sin 2t)^2}$$

$$= \sqrt{9 \sin^2 t \cos^2 t (\cos^2 t + \sin^2 t) + 4 \sin^2 2t}$$

$$= \sqrt{\frac{9}{4} (2 \sin t \cos t)^2 + 4 \sin^2 2t} = \sqrt{\left(\frac{9}{4} + 4\right) \sin^2 2t}$$

$$= \frac{5}{2} \sin 2t$$

Then for $0 \leq t \leq \frac{\pi}{2},$

$$s(t) = \int_0^t |\mathbf{r}'(u)| du = \frac{5}{2} \int_0^t \sin 2u du$$

$$= -\frac{5}{4} [\cos 2u]_0^t = \frac{5}{4} (1 - \cos 2t)$$

Therefore, $t(s) = \frac{1}{2} \cos^{-1} \left(1 - \frac{4}{5}s \right)$ and

$$\mathbf{r}(t(s)) = \cos^3 \left[\frac{1}{2} \cos^{-1} \left(1 - \frac{4}{5}s \right) \right] \mathbf{i} \\ + \sin^3 \left[\frac{1}{2} \cos^{-1} \left(1 - \frac{4}{5}s \right) \right] \mathbf{j} \\ + \cos \left[2 \cdot \frac{1}{2} \cos^{-1} \left(1 - \frac{4}{5}s \right) \right] \mathbf{k} \\ = \cos^3 \left[\frac{1}{2} \cos^{-1} \left(1 - \frac{4}{5}s \right) \right] \mathbf{i} \\ + \sin^3 \left[\frac{1}{2} \cos^{-1} \left(1 - \frac{4}{5}s \right) \right] \mathbf{j} + \left[1 - \frac{4}{5}s \right] \mathbf{k}$$

9. (a) $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{16+9}} \langle 4 \cos 4t, 3, -4 \sin 4t \rangle$

$$= \frac{1}{5} \langle 4 \cos 4t, 3, -4 \sin 4t \rangle$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{5}{16 \cdot 5} \langle -16 \sin 4t, 0, -16 \cos 4t \rangle$$

$$= \langle -\sin 4t, 0, -\cos 4t \rangle$$

(b) $\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{16}{5 \cdot 5} = \frac{16}{25}$

10. (a) $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{6(1+t^2)} \langle 6, 6\sqrt{2}t, 6t^2 \rangle$

$$= \frac{1}{1+t^2} \langle 1, \sqrt{2}t, t^2 \rangle$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

$$= \frac{1}{|\mathbf{T}'(t)|} \left[-\frac{2t}{(1+t^2)^2} \langle 1, \sqrt{2}t, t^2 \rangle \right.$$

$$\left. + \frac{1}{1+t^2} \langle 0, \sqrt{2}, 2t \rangle \right]$$

$$= \frac{1+t^2}{\sqrt{2}} \left[\frac{1}{(1+t^2)^2} \langle -2t, \sqrt{2}(1-t^2), 2t \rangle \right]$$

$$= \frac{1}{\sqrt{2}(1+t^2)} \langle -2t, \sqrt{2}(1-t^2), 2t \rangle$$

(b) $\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{\sqrt{2}}{1+t^2} \cdot \frac{1}{6(1+t^2)}$

$$= \frac{1}{3\sqrt{2}(1+t^2)^2}$$

$$\begin{aligned}
 11. \text{ (a) } \mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \\
 &= \frac{1}{\sqrt{2 \sin^2 t + 2 \cos^2 t}} \langle -\sqrt{2} \sin t, \cos t, \cos t \rangle \\
 &= \frac{1}{\sqrt{2}} \langle -\sqrt{2} \sin t, \cos t, \cos t \rangle \\
 \mathbf{N}(t) &= \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} \\
 &= \frac{1}{\sqrt{2 \cos^2 t + 2 \sin^2 t}} \langle -\sqrt{2} \cos t, -\sin t, -\sin t \rangle \\
 &= \frac{1}{\sqrt{2}} \langle -\sqrt{2} \cos t, -\sin t, -\sin t \rangle
 \end{aligned}$$

$$(b) \kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{2}}$$

$$\begin{aligned}
 12. \text{ (a) } \mathbf{r}'(t) &= \langle t^2, 2t, 2 \rangle \Rightarrow \\
 |\mathbf{r}'(t)| &= \sqrt{t^4 + 4t^2 + 4} = \sqrt{(t^2 + 2)^2} = t^2 + 2. \text{ Then} \\
 \mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{t^2 + 2} \langle t^2, 2t, 2 \rangle. \\
 \mathbf{T}'(t) &= \frac{-2t}{(t^2 + 2)^2} \langle t^2, 2t, 2 \rangle + \frac{1}{t^2 + 2} \langle 2t, 2, 0 \rangle \\
 &\quad \text{(by Theorem 13.2.3 \#3)} \\
 &= \frac{1}{(t^2 + 2)^2} \langle -2t^3, -4t^2, -4t \rangle \\
 &\quad + \frac{1}{(t^2 + 2)^2} \langle 2t^3 + 4t, 2t^2 + 4, 0 \rangle \\
 &= \frac{1}{(t^2 + 2)^2} \langle 4t, 4 - 2t^2, -4t \rangle \\
 |\mathbf{T}'(t)| &= \frac{1}{(t^2 + 2)^2} \sqrt{16t^2 + (16 - 16t^2 + 4t^4) + 16t^2} \\
 &= \frac{1}{(t^2 + 2)^2} \sqrt{4t^4 + 16t^2 + 16} \\
 &= \frac{1}{(t^2 + 2)^2} \sqrt{4(t^2 + 2)^2} = \frac{2(t^2 + 2)}{(t^2 + 2)^2} \\
 &= \frac{2}{t^2 + 2}
 \end{aligned}$$

Thus

$$\begin{aligned}
 \mathbf{N}(t) &= \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{1/(t^2 + 2)^2}{2/(t^2 + 2)} \langle 4t, 4 - 2t^2, -4t \rangle \\
 &= \frac{1}{t^2 + 2} \langle 2t, 2 - t^2, -2t \rangle
 \end{aligned}$$

$$(b) \kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{2/(t^2 + 2)}{t^2 + 2} = \frac{2}{(t^2 + 2)^2}$$

$$\begin{aligned}
 13. \text{ (a) } \mathbf{T}(t) &= \frac{1}{\sqrt{2 + e^{2t} + e^{-2t}}} \langle \sqrt{2}, e^t, -e^{-t} \rangle \\
 &= \frac{1}{e^t + e^{-t}} \langle \sqrt{2}, e^t, -e^{-t} \rangle \\
 &= \frac{1}{e^{2t} + 1} \langle \sqrt{2}e^{2t}, e^{2t}, -1 \rangle
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{T}'(t) &= \frac{-2e^{2t}}{(e^{2t} + 1)^2} \langle \sqrt{2}e^t, e^{2t}, -1 \rangle \\
 &\quad + \frac{1}{e^{2t} + 1} \langle \sqrt{2}e^t, 2e^{2t}, 0 \rangle \\
 &= \frac{1}{(e^{2t} + 1)^2} \langle -2\sqrt{2}e^{3t} + \sqrt{2}e^{3t} + \sqrt{2}e^t, \\
 &\quad -2e^{4t} + 2e^{4t} + 2e^{2t}, 2e^{2t} \rangle
 \end{aligned}$$

$$= \frac{1}{(e^{2t} + 1)^2} \langle \sqrt{2}(e^t - e^{3t}), 2e^{2t}, 2e^{2t} \rangle$$

$$\begin{aligned}
 |\mathbf{T}'(t)| &= \frac{1}{(e^{2t} + 1)^2} \sqrt{2(e^{2t} + e^{6t} - 2e^{4t}) + 8e^{4t}} \\
 &= \frac{\sqrt{2}e^t}{(e^{2t} + 1)^2} (e^{2t} + 1) = \frac{\sqrt{2}e^t}{e^{2t} + 1}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{N}(t) &= \frac{1}{(e^{2t} + 1)^2} \frac{e^{2t} + 1}{\sqrt{2}e^t} \langle \sqrt{2}(e^t - e^{3t}), 2e^{2t}, 2e^{2t} \rangle \\
 &= \frac{1}{\sqrt{2}(e^{2t} + 1)e^t} \langle \sqrt{2}(e^t - e^{3t}), 2e^{2t}, 2e^{2t} \rangle \\
 &= \frac{1}{e^{2t} + 1} \langle 1 - e^{2t}, \sqrt{2}e^t, \sqrt{2}e^t \rangle
 \end{aligned}$$

$$(b) \kappa(t) = \frac{\sqrt{2}e^t}{(e^{2t} + 1)} \cdot \frac{1}{e^t + e^{-t}} = \frac{\sqrt{2}e^{2t}}{(e^{2t} + 1)^2}$$

$$\begin{aligned}
 14. \text{ (a) } \mathbf{T}(t) &= \frac{1}{\sqrt{4t^2 + 4t^4 + 1}} \langle 2t, 2t^2, 1 \rangle \\
 &= \frac{1}{2t^2 + 1} \langle 2t, 2t^2, 1 \rangle \\
 \mathbf{T}'(t) &= -(2t^2 + 1)^{-2} (4t) \langle 2t, 2t^2, 1 \rangle \\
 &\quad + (2t^2 + 1)^{-1} \langle 2, 4t, 0 \rangle \\
 &= \frac{1}{(2t^2 + 1)^2} \langle -8t^2 + 4t^2 + 2, \\
 &\quad -8t^3 + 8t^3 + 4, -4t \rangle
 \end{aligned}$$

$$= \frac{1}{(2t^2 + 1)^2} \langle 1 - 2t^2, 2t, -2t \rangle$$

$$|\mathbf{T}'(t)| = \frac{1}{(2t^2 + 1)^2} \sqrt{1 - 4t^2 + 4t^4 + 8t^2} = \frac{2}{2t^2 + 1}$$

$$\mathbf{N}(t) = \frac{2}{(2t^2 + 1)^2} \cdot \frac{2t^2 + 1}{2} \langle 1 - 2t^2, 2t, -2t \rangle$$

$$= \frac{1}{2t^2 + 1} \langle 1 - 2t^2, 2t, -2t \rangle$$

$$(b) \kappa(t) = \frac{2}{2t^2 + 1} \cdot \frac{1}{2t^2 + 1} = \frac{2}{(2t^2 + 1)^2}$$

$$\begin{aligned}
 15. \mathbf{r}'(t) &= \mathbf{j} - 2t \mathbf{k}, \mathbf{r}''(t) = -2 \mathbf{k}, |\mathbf{r}'(t)|^3 = (4t^2 + 1)^{3/2}, \\
 |\mathbf{r}'(t) \times \mathbf{r}''(t)| &= |-2\mathbf{i}| = 2, \\
 \kappa(t) &= \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{2}{(4t^2 + 1)^{3/2}}
 \end{aligned}$$

$$\begin{aligned}
 16. \mathbf{r}'(t) &= \langle 1, -1, 6t \rangle, \mathbf{r}''(t) = \langle 0, 0, 6 \rangle, \\
 |\mathbf{r}'(t)|^3 &= (\sqrt{2 + 36t^2})^3 = [2(1 + 18t^2)]^{3/2}, \\
 |\mathbf{r}'(t) \times \mathbf{r}''(t)| &= |(-6, -6, 0)| = 6\sqrt{2}, \\
 \kappa(t) &= \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{6\sqrt{2}}{[2(1 + 18t^2)]^{3/2}} = \frac{3}{(1 + 18t^2)^{3/2}}
 \end{aligned}$$

$$\begin{aligned}
17. \mathbf{r}'(t) &= \langle 6t^2, -6t, 6 \rangle, \quad \mathbf{r}''(t) = \langle 12t, -6, 0 \rangle, \\
|\mathbf{r}'(t)|^3 &= 6^3 (t^4 + t^2 + 1)^{3/2}, \\
|\mathbf{r}'(t) \times \mathbf{r}''(t)| &= |36 \langle 1, 2t, t^2 \rangle| = 36\sqrt{1+4t^2+t^4}, \\
\kappa(t) &= \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{36\sqrt{1+4t^2+t^4}}{6^3 (t^4 + t^2 + 1)^{3/2}} \\
&= \frac{\sqrt{1+4t^2+t^4}}{6(t^4 + t^2 + 1)^{3/2}}
\end{aligned}$$

$$\begin{aligned}
18. \mathbf{r}'(t) &= \langle 2t, 2t-4, 2 \rangle, \quad \mathbf{r}''(t) = \langle 2, 2, 0 \rangle, \\
|\mathbf{r}'(t)|^3 &= (4t^2 + 4t^2 - 16t + 16 + 4)^{3/2} \\
&= 8(2t^2 - 4t + 5)^{3/2}, \\
|\mathbf{r}'(t) \times \mathbf{r}''(t)| &= 4|\langle -1, 1, 2 \rangle| = 4\sqrt{6}, \\
\kappa(t) &= \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{4\sqrt{6}}{8(2t^2 - 4t + 5)^{3/2}} \\
&= \frac{\sqrt{6}}{2(2t^2 - 4t + 5)^{3/2}}
\end{aligned}$$

$$\begin{aligned}
19. \mathbf{r}'(t) &= \langle \cos t, -\sin t, \cos t \rangle, \\
\mathbf{r}''(t) &= \langle -\sin t, -\cos t, -\sin t \rangle, \\
|\mathbf{r}'(t)|^3 &= (\sqrt{\cos^2 t + 1})^3, \\
|\mathbf{r}'(t) \times \mathbf{r}''(t)| &= |\langle 1, 0, -1 \rangle| = \sqrt{2}, \\
\kappa(t) &= \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{\sqrt{2}}{(1 + \cos^2 t)^{3/2}}
\end{aligned}$$

$$\begin{aligned}
20. \mathbf{r}'(t) &= \langle \sqrt{2}, e^t, -e^{-t} \rangle. \text{ The point } (0, 1, 1) \\
&\text{ corresponds to } t = 0, \text{ and } \mathbf{r}'(0) = \langle \sqrt{2}, 1, -1 \rangle \\
\Rightarrow |\mathbf{r}'(0)| &= \sqrt{(\sqrt{2})^2 + 1^2 + (-1)^2} = 2. \\
\mathbf{r}''(t) &= \langle 0, e^t, e^{-t} \rangle \Rightarrow \mathbf{r}''(0) = \langle 0, 1, 1 \rangle. \\
\mathbf{r}'(0) \times \mathbf{r}''(0) &= \langle 2, -\sqrt{2}, \sqrt{2} \rangle, \\
|\mathbf{r}'(0) \times \mathbf{r}''(0)| &= \sqrt{2^2 + (-\sqrt{2})^2 + (\sqrt{2})^2} \\
&= \sqrt{8} = 2\sqrt{2} \\
\text{Then } \kappa(0) &= \frac{|\mathbf{r}'(0) \times \mathbf{r}''(0)|}{|\mathbf{r}'(0)|^3} = \frac{2\sqrt{2}}{2^3} = \frac{\sqrt{2}}{4}.
\end{aligned}$$

$$\begin{aligned}
21. y' &= \frac{1}{2\sqrt{x}}, \quad y'' = -\frac{1}{4(x)^{3/2}}, \\
\kappa(x) &= \frac{|y''(x)|}{[1 + (y'(x))^2]^{3/2}} = \frac{1}{4|x^{3/2}|} \frac{1}{[1 + 1/(4x)]^{3/2}} \\
&= \frac{2}{(4x+1)^{3/2}}
\end{aligned}$$

$$\begin{aligned}
22. y' &= \cos x, \quad y'' = -\sin x, \\
\kappa(x) &= \frac{|y''(x)|}{[1 + (y'(x))^2]^{3/2}} = \frac{|\sin x|}{(1 + \cos^2 x)^{3/2}}
\end{aligned}$$

$$\begin{aligned}
23. y' &= \frac{1}{x}, \quad y'' = -\frac{1}{x^2}, \\
\kappa(x) &= \frac{|y''(x)|}{[1 + (y'(x))^2]^{3/2}} = \left| \frac{-1}{x^2} \right| \frac{1}{(1 + 1/x^2)^{3/2}} \\
&= \frac{1}{x^2} \frac{(x^2)^{3/2}}{(x^2 + 1)^{3/2}} = \frac{|x|}{(x^2 + 1)^{3/2}}
\end{aligned}$$

$$\begin{aligned}
24. \kappa(t) &= \frac{|\dot{x}\ddot{y} - \ddot{x}y|}{(\dot{x}^2 + \dot{y}^2)^{3/2}} = \frac{|(3t^2)(2) - (6t)(2t)|}{(9t^4 + 4t^2)^{3/2}} \\
&= \frac{6t^2}{(t^2)^{3/2} (9t^2 + 4)^{3/2}} = \frac{6t^2}{|t|^3 (9t^2 + 4)^{3/2}} \\
&= \frac{6}{|t| (9t^2 + 4)^{3/2}}
\end{aligned}$$

$$\begin{aligned}
25. \kappa(t) &= \frac{|\dot{x}\ddot{y} - \ddot{x}y|}{(\dot{x}^2 + \dot{y}^2)^{3/2}} \\
&= \frac{|(\sin t + t \cos t)(-2 \sin t - t \cos t) - (2 \cos t - t \sin t)(\cos t - t \sin t)|}{(\dot{x}^2 + \dot{y}^2)^{3/2}} \\
&= \frac{|(-2 \sin^2 t - 3t \sin t \cos t - t^2 \cos^2 t) - (2 \cos^2 t - 3t \cos t \sin t + t^2 \sin^2 t)|}{(\dot{x}^2 + \dot{y}^2)^{3/2}} \\
&= \frac{|-(\sin^2 t + \cos^2 t)(2 + t^2)|}{\left(\sin^2 t + 2t \cos t \sin t + \cos^2 t + t^2 \cos^2 t - 2t \sin t \cos t + t^2 \sin t \right)^{3/2}} \\
&= \frac{2 + t^2}{(1 + t^2)^{3/2}}
\end{aligned}$$