10.2 Derivatives and Integrals of Vector Functions

A Click here for answers.

S Click here for solutions.

1–3
(a) Sketch the plane curve with the given vector equation.
(b) Find $\mathbf{r}'(t)$.
(c) Sketch the position vector $\mathbf{r}(t)$ and the tangent vector $\mathbf{r}'(t)$ for the given value of t.

1. $\mathbf{r}(t) = \langle t^3, t^2 \rangle, \quad t = 1$

2. $\mathbf{r}(t) = e^t\,\mathbf{i} + e^{-2t}\,\mathbf{j}, \quad t = 0$

3. $\mathbf{r}(t) = \sec t\,\mathbf{i} + \tan t\,\mathbf{j}, \quad t = \pi/4$

4–7 Find the domain and derivative of the vector function.

4. $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$

5. $\mathbf{r}(t) = \left\langle t^2 - 4, \sqrt{t-4}, \sqrt{6-t} \right\rangle$

6. $\mathbf{r}(t) = \mathbf{i} + \tan t\,\mathbf{j} + \sec t\,\mathbf{k}$

7. $\mathbf{r}(t) = te^{2t}\,\mathbf{i} + \dfrac{t-1}{t+1}\,\mathbf{j} + \tan^{-1}t\,\mathbf{k}$

8–9 Find the derivative of the vector function.

8. $\mathbf{r}(t) = \ln(4 - t^2)\,\mathbf{i} + \sqrt{1+t}\,\mathbf{j} - 4e^{3t}\,\mathbf{k}$

9. $\mathbf{r}(t) = e^{-t}\cos t\,\mathbf{i} + e^{-t}\sin t\,\mathbf{j} + \ln|t|\,\mathbf{k}$

10–14 Find the unit tangent vector $\mathbf{T}(t)$ at the point with the given value of the parameter t.

10. $\mathbf{r}(t) = \left\langle \sqrt{t}, t - t^2, \tan^{-1}t \right\rangle, \quad t = 1$

11. $\mathbf{r}(t) = t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 3\cos t\,\mathbf{k}, \quad t = \pi/6$

12. $\mathbf{r}(t) = e^{2t}\cos t\,\mathbf{i} + e^{2t}\sin t\,\mathbf{j} + e^{2t}\,\mathbf{k}, \quad t = \pi/2$

13. $\mathbf{r}(t) = \langle 2t, 3t^2, 4t^3 \rangle, \quad t = 1$

14. $\mathbf{r}(t) = \langle e^{2t}, e^{-2t}, te^{2t} \rangle, \quad t = 0$

15–20 Find parametric equations for the tangent line to the curve with the given parametric equations at the specified point.

15. $x = t,\ y = t^2,\ z = t^3;\quad (1, 1, 1)$

16. $x = 1 + 2t,\ y = 1 + t - t^2,\ z = 1 - t + t^2 - t^3;\quad (1, 1, 1)$

17. $x = t\cos 2\pi t,\ y = t\sin 2\pi t,\ z = 4t;\quad \left(0, \frac{1}{4}, 1\right)$

18. $x = \sin \pi t,\ y = \sqrt{t},\ z = \cos \pi t;\quad (0, 1, -1)$

19. $x = t,\ y = \sqrt{2}\cos t,\ z = \sqrt{2}\sin t;\quad (\pi/4, 1, 1)$

20. $x = \cos t,\ y = 3e^{2t},\ z = 3e^{-2t};\quad (1, 3, 3)$

21–23 Evaluate the integral.

21. $\displaystyle\int_0^1 (t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k})\,dt$

22. $\displaystyle\int_1^2 [(1 + t^2)\,\mathbf{i} - 4t^4\,\mathbf{j} - (t^2 - 1)\,\mathbf{k}]\,dt$

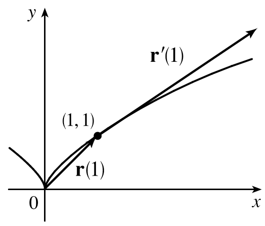
23. $\displaystyle\int_0^{\pi/4} (\cos 2t\,\mathbf{i} + \sin 2t\,\mathbf{j} + t\sin t\,\mathbf{k})\,dt$

Answers

E Click here for exercises.

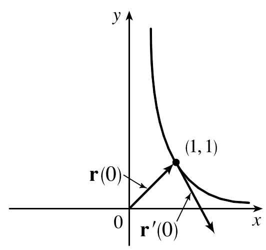
S Click here for solutions.

1. (a), (c)



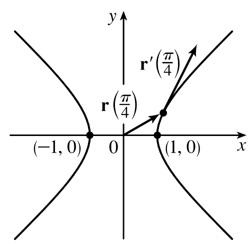
(b) $\langle 3t^2, 2t\rangle$

2. (a), (c)



(b) $e^t\,\mathbf{i} - 2e^{-2t}\,\mathbf{j}$

3. (a), (c)



(b) $\sec t\tan t\,\mathbf{i} + \sec^2 t\,\mathbf{j}$

4. $\mathbb{R}$, $\mathbf{r}'(t) = \langle 1, 2t, 3t^2\rangle$

5. $\{t \mid 4 \le t \le 6\}$, $\mathbf{r}'(t) = \left\langle 2t, \dfrac{1}{2\sqrt{t-4}}, -\dfrac{1}{2\sqrt{6-t}}\right\rangle$

6. $\left\{t \mid t \neq (2n+1)\frac{\pi}{2}, n \text{ an integer}\right\}$,
$\mathbf{r}'(t) = \left(\sec^2 t\right)\mathbf{j} + (\sec t\tan t)\,\mathbf{k}$

7. $\{t \mid t \neq -1\}$, $\mathbf{r}'(t) = (1+2t)\,e^{2t}\,\mathbf{i} + \dfrac{2}{(t+1)^2}\,\mathbf{j} + \dfrac{1}{1+t^2}\,\mathbf{k}$

8. $\mathbf{r}'(t) = -\dfrac{2t}{4-t^2}\,\mathbf{i} + \dfrac{1}{2\sqrt{1+t}}\,\mathbf{j} - 12e^{3t}\,\mathbf{k}$

9. $\mathbf{r}'(t) = -e^{-t}(\cos t + \sin t)\,\mathbf{i} + e^{-t}(\cos t - \sin t)\,\mathbf{j} + \dfrac{1}{t}\,\mathbf{k}$

10. $\left\langle \frac{1}{\sqrt{6}}, -\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{6}}\right\rangle$

11. $\frac{2}{5}\,\mathbf{i} + \frac{2\sqrt{3}}{5}\,\mathbf{j} - \frac{3}{5}\,\mathbf{k}$

12. $-\frac{1}{3}\,\mathbf{i} + \frac{2}{3}\,\mathbf{j} + \frac{2}{3}\,\mathbf{k}$

13. $\left\langle \frac{1}{\sqrt{46}}, \frac{3}{\sqrt{46}}, \frac{6}{\sqrt{46}}\right\rangle$

14. $\left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right\rangle$

15. $x = 1+t,\ y = 1+2t,\ z = 1+3t$

16. $x = 1+2t,\ y = 1+t,\ z = 1-t$

17. $x = -\frac{\pi}{2}t,\ y = \frac{1}{4}+t,\ z = 1+4t$

18. $x = -\pi t,\ y = 1+\frac{1}{2}t,\ z = -1$

19. $x = \frac{\pi}{4}+t,\ y = 1-t,\ z = 1+t$

20. $x = 1,\ y = 3+6t,\ z = 3-6t$

21. $\frac{1}{2}\,\mathbf{i} + \frac{1}{3}\,\mathbf{j} + \frac{1}{4}\,\mathbf{k}$

22. $\frac{10}{3}\,\mathbf{i} - \frac{124}{5}\,\mathbf{j} - \frac{4}{3}\,\mathbf{k}$

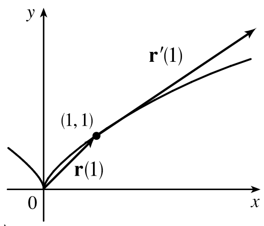
23. $\frac{1}{2}\,\mathbf{i} + \frac{1}{2}\,\mathbf{j} + \frac{4-\pi}{4\sqrt{2}}\,\mathbf{k}$

Solutions

E Click here for exercises.

A Click here for answers.

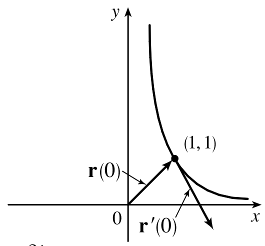
1. (a), (c)



(b) $\mathbf{r}'(t) = \langle 3t^2, 2t \rangle$

2. $x^{-2} = e^{-2t} = y$, so $y = 1/x^2$, $x > 0$.

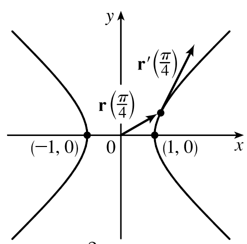
(a), (c)



(b) $\mathbf{r}'(t) = e^t\,\mathbf{i} - 2e^{-2t}\,\mathbf{j}$

3. $x^2 - y^2 = \sec^2 t - \tan^2 t = 1$, so the curve is a hyperbola.

(a), (c)



(b) $\mathbf{r}'(t) = \sec t \tan t\,\mathbf{i} + \sec^2 t\,\mathbf{j}$

4. The domain of $\mathbf{r}$ is $\mathbb{R}$ and $\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle$.

5. The domain of $\mathbf{r}$ is $\{t \mid t \geq 4 \text{ and } t \leq 6\}$ or $\{t \mid 4 \leq t \leq 6\}$ and

$$\mathbf{r}'(t) = \left\langle 2t, \tfrac{1}{2}(t-4)^{-1/2}, \tfrac{1}{2}(6-t)^{-1/2}(-1) \right\rangle = \left\langle 2t, \frac{1}{2\sqrt{t-4}}, -\frac{1}{2\sqrt{6-t}} \right\rangle$$

6. Since $\tan t$ and $\sec t$ are not defined for odd multiples of $\frac{\pi}{2}$, the domain of $\mathbf{r}$ is $\left\{t \mid t \neq (2n+1)\frac{\pi}{2}, n \text{ an integer}\right\}$.

$\mathbf{r}'(t) = \left(\sec^2 t\right)\mathbf{j} + (\sec t \tan t)\,\mathbf{k}$.

7. Since $\dfrac{t-1}{t+1}$ is not defined for $t = -1$ (and $\tan^{-1} t$ is defined for all real t), the domain is $\{t \mid t \neq -1\}$.

$$\mathbf{r}'(t) = (1+2t)\,e^{2t}\,\mathbf{i} + \frac{2}{(t+1)^2}\,\mathbf{j} + \frac{1}{1+t^2}\,\mathbf{k}.$$

8. $\mathbf{r}'(t) = -\dfrac{2t}{4-t^2}\,\mathbf{i} + \dfrac{1}{2\sqrt{1+t}}\,\mathbf{j} - 12e^{3t}\,\mathbf{k}$

9. $\mathbf{r}'(t) = -e^{-t}(\cos t + \sin t)\,\mathbf{i} + e^{-t}(\cos t - \sin t)\,\mathbf{j} + \dfrac{1}{t}\,\mathbf{k}$

10. $\mathbf{r}'(t) = \left\langle \dfrac{1}{2\sqrt{t}}, 1-2t, \dfrac{1}{1+t^2} \right\rangle \Rightarrow$

$\mathbf{r}'(1) = \left\langle \frac{1}{2}, -1, \frac{1}{2} \right\rangle$. Thus

$|\mathbf{r}'(1)| = \sqrt{\left(\frac{1}{2}\right)^2 + (-1)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{2}}$ and

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{|\mathbf{r}'(1)|} = \frac{1}{\sqrt{3/2}}\left\langle \tfrac{1}{2}, -1, \tfrac{1}{2} \right\rangle = \left\langle \tfrac{1}{2}\sqrt{\tfrac{2}{3}}, -\sqrt{\tfrac{2}{3}}, \tfrac{1}{2}\sqrt{\tfrac{2}{3}} \right\rangle = \left\langle \tfrac{1}{\sqrt{6}}, -\sqrt{\tfrac{2}{3}}, \tfrac{1}{\sqrt{6}} \right\rangle$$

11. $\mathbf{r}'(t) = \mathbf{i} + 2\cos t\,\mathbf{j} - 3\sin t\,\mathbf{k}$, $\mathbf{r}'\left(\frac{\pi}{6}\right) = \mathbf{i} + \sqrt{3}\,\mathbf{j} - \frac{3}{2}\,\mathbf{k}$. Thus

$$\mathbf{T}\left(\tfrac{\pi}{6}\right) = \frac{1}{\sqrt{1^2 + \left(\sqrt{3}\right)^2 + (-3/2)^2}}\left(\mathbf{i} + \sqrt{3}\,\mathbf{j} - \tfrac{3}{2}\,\mathbf{k}\right) = \frac{1}{5/2}\left(\mathbf{i} + \sqrt{3}\,\mathbf{j} - \tfrac{3}{2}\,\mathbf{k}\right) = \tfrac{2}{5}\,\mathbf{i} + \tfrac{2\sqrt{3}}{5}\,\mathbf{j} - \tfrac{3}{5}\,\mathbf{k}$$

12. $\mathbf{r}'(t) = 2e^{2t}(\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \mathbf{k}) + e^{2t}(-\sin t\,\mathbf{i} + \cos t\,\mathbf{j})$
$= e^{2t}\left[(2\cos t - \sin t)\,\mathbf{i} + (2\sin t + \cos t)\,\mathbf{j} + 2\,\mathbf{k}\right]$

$\mathbf{r}'\left(\frac{\pi}{2}\right) = e^{\pi}(-\mathbf{i} + 2\,\mathbf{j} + 2\,\mathbf{k})$

Thus, $\mathbf{T}\left(\frac{\pi}{2}\right) = \dfrac{e^{\pi}}{e^{\pi}\sqrt{9}}(-\mathbf{i} + 2\,\mathbf{j} + 2\,\mathbf{k}) = -\frac{1}{3}\,\mathbf{i} + \frac{2}{3}\,\mathbf{j} + \frac{2}{3}\,\mathbf{k}$.

13. $\mathbf{r}'(t) = \langle 2, 6t, 12t^2 \rangle$, $\mathbf{r}(1) = \langle 2, 3, 4 \rangle$, $\mathbf{r}'(1) = \langle 2, 6, 12 \rangle$. Thus,

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{|\mathbf{r}'(1)|} = \frac{1}{\sqrt{188}}\langle 2, 6, 12 \rangle = \left\langle \tfrac{1}{\sqrt{46}}, \tfrac{3}{\sqrt{46}}, \tfrac{6}{\sqrt{46}} \right\rangle.$$

14. $\mathbf{r}'(t) = \left\langle 2e^{2t}, -2e^{-2t}, (1+2t)\,e^{2t} \right\rangle$, $\mathbf{r}'(0) = \langle 2, -2, 1 \rangle$.

Thus, $\mathbf{T}(0) = \frac{1}{\sqrt{9}}\langle 2, -2, 1 \rangle = \left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right\rangle$.

15. The vector equation of the curve is $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$, so $\mathbf{r}'(t) = \mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k}$. At the point $(1,1,1)$, $t = 1$, so the tangent vector here is $\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$. The tangent line goes through the point $(1,1,1)$ and has direction vector $\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$. Thus parametric equations are $x = 1 + t$, $y = 1 + 2t$, $z = 1 + 3t$.

16. $\mathbf{r}(t) = \left\langle 1 + 2t, 1 + t - t^2, 1 - t + t^2 - t^3 \right\rangle$,

$\mathbf{r}'(t) = \left\langle 2, 1 - 2t, -1 + 2t - 3t^2 \right\rangle$. At $(1,1,1)$, $t = 0$ and $\mathbf{r}'(0) = \langle 2, 1, -1 \rangle$. Thus the tangent lines goes through the point $(1,1,1)$ and has direction vector $\langle 2, 1, -1 \rangle$. The parametric equations are $x = 1 + 2t$, $y = 1 + t$, $z = 1 - t$.

17. $\mathbf{r}(t) = \langle t\cos 2\pi t, t\sin 2\pi t, 4t \rangle$,

$\mathbf{r}'(t) = \langle \cos 2\pi t - 2\pi t \sin 2\pi t, \sin 2\pi t + 2\pi t \cos 2\pi t, 4 \rangle$.

At $\left(0, \frac{1}{4}, 1\right)$, $t = \frac{1}{4}$ and

$\mathbf{r}'\left(\frac{1}{4}\right) = \left\langle 0 - \frac{\pi}{2}, 1 + 0, 4 \right\rangle = \left\langle -\frac{\pi}{2}, 1, 4 \right\rangle$. Thus, parametric equations of the tangent line are $x = -\frac{\pi}{2}t$, $y = \frac{1}{4} + t$, $z = 1 + 4t$.

18. $\mathbf{r}(t) = \left\langle \sin \pi t, \sqrt{t}, \cos \pi t \right\rangle$,
$\mathbf{r}'(t) = \left\langle \pi \cos \pi t, 1/\left(2\sqrt{t}\right), -\pi \sin \pi t \right\rangle$. At $(0, 1, -1)$, $t = 1$ and $\mathbf{r}'(1) = \left\langle -\pi, \frac{1}{2}, 0 \right\rangle$. Thus, parametric equations of the tangent line are $x = -\pi t$, $y = 1 + \frac{1}{2}t$, $z = -1$.

19. $\mathbf{r}(t) = \left\langle t, \sqrt{2}\cos t, \sqrt{2}\sin t \right\rangle$,
$\mathbf{r}'(t) = \left\langle 1, -\sqrt{2}\sin t, \sqrt{2}\cos t \right\rangle$. At $\left(\frac{\pi}{4}, 1, 1\right)$, $t = \frac{\pi}{4}$ and $\mathbf{r}'\left(\frac{\pi}{4}\right) = \langle 1, -1, 1 \rangle$. Thus, parametric equations of the tangent line are $x = \frac{\pi}{4} + t$, $y = 1 - t$, $z = 1 + t$.

20. $\mathbf{r}(t) = \left\langle \cos t, 3e^{2t}, 3e^{-2t} \right\rangle$,
$\mathbf{r}'(t) = \left\langle -\sin t, 6e^{2t}, -6e^{-2t} \right\rangle$. At $(1, 3, 3)$, $t = 0$ and $\mathbf{r}'(0) = \langle 0, 6, -6 \rangle$. Thus, parametric equations of the tangent line are $x = 1$, $y = 3 + 6t$, $z = 3 - 6t$.

21.
$$\begin{aligned}\int_0^1 \left(t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}\right) dt &= \left(\int_0^1 t\,dt\right)\mathbf{i} + \left(\int_0^1 t^2\,dt\right)\mathbf{j} + \left(\int_0^1 t^3\,dt\right)\mathbf{k} \\ &= \left[\frac{t^2}{2}\right]_0^1 \mathbf{i} + \left[\frac{t^3}{3}\right]_0^1 \mathbf{j} + \left[\frac{t^4}{4}\right]_0^1 \mathbf{k} \\ &= \tfrac{1}{2}\,\mathbf{i} + \tfrac{1}{3}\,\mathbf{j} + \tfrac{1}{4}\,\mathbf{k}\end{aligned}$$

22.
$$\begin{aligned}\int_1^2 \left[\left(1 + t^2\right)\mathbf{i} - 4t^4\,\mathbf{j} - \left(t^2 - 1\right)\mathbf{k}\right] dt &= \left[\left(t + \tfrac{1}{3}t^3\right)\mathbf{i} - \tfrac{4}{5}t^5\,\mathbf{j} - \left(\tfrac{1}{3}t^3 - t\right)\mathbf{k}\right]_1^2 \\ &= \left[\left(2 + \tfrac{8}{3}\right)\mathbf{i} - \tfrac{128}{5}\,\mathbf{j} - \left(\tfrac{8}{3} - 2\right)\mathbf{k}\right] \\ &\qquad - \left[\left(1 + \tfrac{1}{3}\right)\mathbf{i} - \tfrac{4}{5}\,\mathbf{j} - \left(\tfrac{1}{3} - 1\right)\mathbf{k}\right] \\ &= \tfrac{10}{3}\,\mathbf{i} - \tfrac{124}{5}\,\mathbf{j} - \tfrac{4}{3}\,\mathbf{k}\end{aligned}$$

23.
$$\begin{aligned}\int_0^{\pi/4} \left(\cos 2t\,\mathbf{i} + \sin 2t\,\mathbf{j} + t\sin t\,\mathbf{k}\right) dt &= \left[\tfrac{1}{2}\sin 2t\,\mathbf{i} - \tfrac{1}{2}\cos 2t\,\mathbf{j}\right]_0^{\pi/4} \\ &\qquad + \left[\left[-t\cos t\right]_0^{\pi/4} + \int_0^{\pi/4} \cos t\,dt\right]\mathbf{k} \\ &= \tfrac{1}{2}\,\mathbf{i} + \tfrac{1}{2}\,\mathbf{j} + \left[-\tfrac{\pi}{4}\cos\tfrac{\pi}{4} + \sin\tfrac{\pi}{4}\right]\mathbf{k} \\ &= \tfrac{1}{2}\,\mathbf{i} + \tfrac{1}{2}\,\mathbf{j} + \tfrac{1}{\sqrt{2}}\left(1 - \tfrac{\pi}{4}\right)\mathbf{k} \\ &= \tfrac{1}{2}\,\mathbf{i} + \tfrac{1}{2}\,\mathbf{j} + \tfrac{4-\pi}{4\sqrt{2}}\,\mathbf{k}\end{aligned}$$