9.4 The Cross Product

A Click here for answers.

S Click here for solutions.

1–9 Find the cross product $\mathbf{a} \times \mathbf{b}$.

1. $\mathbf{a} = \langle 1, 0, 1 \rangle, \quad \mathbf{b} = \langle 0, 1, 0 \rangle$

2. $\mathbf{a} = \langle 2, 4, 0 \rangle, \quad \mathbf{b} = \langle -3, 1, 6 \rangle$

3. $\mathbf{a} = \langle -2, 3, 4 \rangle, \quad \mathbf{b} = \langle 3, 0, 1 \rangle$

4. $\mathbf{a} = \langle 1, 2, -3 \rangle, \quad \mathbf{b} = \langle 5, -1, -2 \rangle$

5. $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}, \quad \mathbf{b} = \mathbf{i} + \mathbf{j} - \mathbf{k}$

6. $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}, \quad \mathbf{b} = 3\mathbf{i} - \mathbf{j} + 7\mathbf{k}$

7. $\mathbf{a} = 2\mathbf{i} - \mathbf{k}, \quad \mathbf{b} = \mathbf{i} + 2\mathbf{j}$

8. $\mathbf{a} = \langle 1, -1, 0 \rangle, \quad \mathbf{b} = \langle 3, 2, 1 \rangle$

9. $\mathbf{a} = \langle -3, 2, 2 \rangle, \quad \mathbf{b} = \langle 6, 3, 1 \rangle$

10. If $\mathbf{a} = \langle 0, 1, 2 \rangle$ and $\mathbf{b} = \langle 3, 1, 0 \rangle$, find $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$.

11. If $\mathbf{a} = \langle -4, 0, 3 \rangle$, $\mathbf{b} = \langle 2, -1, 0 \rangle$, and $\mathbf{c} = \langle 0, 2, 5 \rangle$, show that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.

12. Find two unit vectors orthogonal to both $\mathbf{i} + \mathbf{j}$ and $\mathbf{i} - \mathbf{j} + \mathbf{k}$.

13. Find the area of the parallelogram with vertices $A(0, 1)$, $B(3, 0)$, $C(5, -2)$, and $D(2, -1)$.

14. Find the area of the parallelogram with vertices $P(0, 0, 0)$, $Q(5, 0, 0)$, $R(2, 6, 6)$, and $S(7, 6, 6)$.

15–17 (a) Find a vector orthogonal to the plane through the points P, Q, and R, and (b) find the area of triangle PQR.

15. $P(1, 0, -1), \quad Q(2, 4, 5), \quad R(3, 1, 7)$

16. $P(0, 0, 0), \quad Q(1, -1, 1), \quad R(4, 3, 7)$

17. $P(-4, -4, -4), \quad Q(0, 5, -1), \quad R(3, 1, 2)$

18–19 Find the volume of the parallelepiped determined by the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

18. $\mathbf{a} = \langle 1, 0, 6 \rangle, \quad \mathbf{b} = \langle 2, 3, -8 \rangle, \quad \mathbf{c} = \langle 8, -5, 6 \rangle$

19. $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}, \quad \mathbf{b} = \mathbf{i} - \mathbf{j}, \quad \mathbf{c} = 2\mathbf{i} + 3\mathbf{k}$

20. Given the points $P(1, 1, 1)$, $Q(2, 0, 3)$, $R(4, 1, 7)$, and $S(3, -1, -2)$, find the volume of the parallelepiped with adjacent edges PQ, PR, and PS.

Answers

E Click here for exercises.

S Click here for solutions.

1. $-\mathbf{i} + \mathbf{k}$
2. $24\mathbf{i} - 12\mathbf{j} + 14\mathbf{k}$
3. $3\mathbf{i} + 14\mathbf{j} - 9\mathbf{k}$
4. $-7\mathbf{i} - 13\mathbf{j} - 11\mathbf{k}$
5. $-2\mathbf{i} + 2\mathbf{j}$
6. $13\mathbf{i} - 10\mathbf{j} - 7\mathbf{k}$
7. $2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$
8. $-\mathbf{i} - \mathbf{j} + 5\mathbf{k}$
9. $-4\mathbf{i} + 15\mathbf{j} - 21\mathbf{k}$
10. $-2\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$, $2\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$
12. $\pm\frac{1}{\sqrt{6}}\langle 1, -1, -2\rangle$
13. 4
14. $30\sqrt{2}$
15. (a) $\langle 26, 4, -7\rangle$ (b) $\frac{1}{2}\sqrt{741}$
16. (a) $\langle -10, -3, 7\rangle$ (b) $\frac{1}{2}\sqrt{158}$
17. (a) $\langle 39, -3, -43\rangle$ (b) $\frac{1}{2}\sqrt{3379}$
18. 226
19. 19
20. 21

Solutions

E Click here for exercises.

A Click here for answers.

1. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\mathbf{k}$

$= -\mathbf{i} + \mathbf{k}$

2. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & 0 \\ -3 & 1 & 6 \end{vmatrix}$

$= \begin{vmatrix} 4 & 0 \\ 1 & 6 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 0 \\ -3 & 6 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 4 \\ -3 & 1 \end{vmatrix}\mathbf{k}$

$= 24\mathbf{i} - 12\mathbf{j} + 14\mathbf{k}$

3. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 3 & 4 \\ 3 & 0 & 1 \end{vmatrix}$

$= \begin{vmatrix} 3 & 4 \\ 0 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -2 & 4 \\ 3 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -2 & 3 \\ 3 & 0 \end{vmatrix}\mathbf{k}$

$= 3\mathbf{i} + 14\mathbf{j} - 9\mathbf{k}$

4. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -3 \\ 5 & -1 & -2 \end{vmatrix}$

$= \begin{vmatrix} 2 & -3 \\ -1 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -3 \\ 5 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ 5 & -1 \end{vmatrix}\mathbf{k}$

$= -7\mathbf{i} - 13\mathbf{j} - 11\mathbf{k}$

5. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$

$= \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}\mathbf{k}$

$= -2\mathbf{i} + 2\mathbf{j}$

6. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 3 & -1 & 7 \end{vmatrix}$

$= \begin{vmatrix} 2 & -1 \\ -1 & 7 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -1 \\ 3 & 7 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix}\mathbf{k}$

$= 13\mathbf{i} - 10\mathbf{j} - 7\mathbf{k}$

7. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 1 & 2 & 0 \end{vmatrix}$

$= \begin{vmatrix} 0 & -1 \\ 2 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix}\mathbf{k}$

$= 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$

8. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 3 & 2 & 1 \end{vmatrix}$

$= \begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix}\mathbf{k}$

$= (-1-0)\,\mathbf{i} - (1-0)\,\mathbf{j} + [2-(-3)]\,\mathbf{k}$

$= -\mathbf{i} - \mathbf{j} + 5\mathbf{k}$

9. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 2 & 2 \\ 6 & 3 & 1 \end{vmatrix}$

$= \begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -3 & 2 \\ 6 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -3 & 2 \\ 6 & 3 \end{vmatrix}\mathbf{k}$

$= (2-6)\,\mathbf{i} - (-3-12)\,\mathbf{j} + (-9-12)\,\mathbf{k}$

$= -4\mathbf{i} + 15\mathbf{j} - 21\mathbf{k}$

10. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 2 \\ 3 & 1 & 0 \end{vmatrix}$

$= \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & 2 \\ 3 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix}\mathbf{k}$

$= -2\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$

$\mathbf{b}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 0 \\ 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix}\mathbf{k}$

$= 2\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$

(and notice $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$ here, as we know is always true by Theorem 8.)

11. $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{a}\times \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 0 \\ 0 & 2 & 5 \end{vmatrix}$

$= \mathbf{a}\times\left[\begin{vmatrix} -1 & 0 \\ 2 & 5 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 0 \\ 0 & 5 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -1 \\ 0 & 2 \end{vmatrix}\mathbf{k}\right]$

$= \mathbf{a}\times(-5\mathbf{i} - 10\mathbf{j} + 4\mathbf{k})$

$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 0 & 3 \\ -5 & -10 & 4 \end{vmatrix}$

$= \begin{vmatrix} 0 & 3 \\ -10 & 4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -4 & 3 \\ -5 & 4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -4 & 0 \\ -5 & -10 \end{vmatrix}\mathbf{k}$

$= 30\mathbf{i} + \mathbf{j} + 40\mathbf{k}$

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 0 & 3 \\ 2 & -1 & 0 \end{vmatrix} \times \mathbf{c}$$

$$= \left[\begin{vmatrix} 0 & 3 \\ -1 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -4 & 3 \\ 2 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -4 & 0 \\ 2 & -1 \end{vmatrix} \mathbf{k} \right] \times \mathbf{c}$$

$$= (3\mathbf{i} + 6\mathbf{j} + 4\mathbf{k}) \times \mathbf{c}$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 6 & 4 \\ 0 & 2 & 5 \end{vmatrix} = \begin{vmatrix} 6 & 4 \\ 2 & 5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & 4 \\ 0 & 5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & 6 \\ 0 & 2 \end{vmatrix} \mathbf{k}$$

$$= 22\mathbf{i} - 15\mathbf{j} + 6\mathbf{k}$$

Thus $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) \neq (\mathbf{a}\times\mathbf{b})\times\mathbf{c}$.

12. We know that the cross product of two vectors is orthogonal to both. So we calculate

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \mathbf{k}$$

$$= \mathbf{i} - \mathbf{j} - 2\mathbf{k}$$

Thus, two unit vectors orthogonal to both are $\pm\frac{1}{\sqrt{6}}\langle 1,-1,-2\rangle$, that is, $\left\langle \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}} \right\rangle$ and $\left\langle -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right\rangle$.

13. We know that the area of the parallelogram determined by two vectors is equal to the length of the cross product of these vectors. The vectors corresponding to $\overrightarrow{AB}$ and $\overrightarrow{AD}$ are $\mathbf{a} = \langle 3,-1,0\rangle$ and $\mathbf{b} = \langle 2,-2,0\rangle$, so the area of parallelogram $ABCD$ is

$$|\mathbf{a}\times\mathbf{b}| = \left\| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 0 \\ 2 & -2 & 0 \end{matrix} \right\|$$

$$= |(0)\,\mathbf{i} - (0)\,\mathbf{j} + (-6+2)\,\mathbf{k}| = |-4\mathbf{k}| = 4$$

14. $\overrightarrow{PQ} = \langle 5,0,0\rangle$ and $\overrightarrow{PR} = \langle 2,6,6\rangle$, so the area of parallelogram $PQRS$ is

$$\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right| = \left\| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 0 & 0 \\ 2 & 6 & 6 \end{matrix} \right\|$$

$$= |(0)\,\mathbf{i} - (30)\,\mathbf{j} + (30)\,\mathbf{k}| = |-30\mathbf{j} + 30\mathbf{k}|$$

$$= 30\sqrt{2}$$

15. (a) $\overrightarrow{PQ} = \langle 1,4,6\rangle$ and $\overrightarrow{PR} = \langle 2,1,8\rangle$, so a vector orthogonal to the plane through P, Q, and R is

$$\overrightarrow{PQ}\times\overrightarrow{PR} = \langle 4\cdot 8 - 6\cdot 1, 6\cdot 2 - 1\cdot 8, 1\cdot 1 - 4\cdot 2\rangle$$

$$= \langle 26, 4, -7\rangle \text{ (or any scalar multiple thereof).}$$

(b) The area of the parallelogram determined by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is

$$\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right| = |\langle 26,4,-7\rangle| = \sqrt{676+16+49}$$

$$= \sqrt{741}$$

so the area of triangle PQR is $\frac{1}{2}\sqrt{741}$.

16. (a) $\overrightarrow{PQ} = \langle 1,-1,1\rangle$ and $\overrightarrow{PR} = \langle 4,3,7\rangle$, so a vector orthogonal to the plane through P, Q, and R is

$$\overrightarrow{PQ}\times\overrightarrow{PR}$$

$$= \langle (-1)\cdot 7 - 1\cdot 3, 1\cdot 4 - 1\cdot 7, 1\cdot 3 - (-1)\cdot 4\rangle$$

$$= \langle -10, -3, 7\rangle \text{ (or any scalar multiple thereof).}$$

(b) The area of the parallelogram determined by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is

$$\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right| = |\langle -10,-3,7\rangle| = \sqrt{100+9+49}$$

$$= \sqrt{158}$$

so the area of triangle PQR is $\frac{1}{2}\sqrt{158}$.

17. (a) $\overrightarrow{PQ} = \langle 4,9,3\rangle$ and $\overrightarrow{PR} = \langle 7,5,6\rangle \Rightarrow$

$$\overrightarrow{PQ}\times\overrightarrow{PR} = \langle 9\cdot 6 - 3\cdot 5, 3\cdot 7 - 4\cdot 6, 4\cdot 5 - 9\cdot 7\rangle$$

$$= \langle 39, -3, -43\rangle$$

(b) $\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right| = \sqrt{1521+9+1849} = \sqrt{3379}$, so the area of the triangle is $\frac{1}{2}\sqrt{3379}$.

18. We know that the volume of the parallelepiped determined by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is the magnitude of their scalar triple product, which is

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} 1 & 0 & 6 \\ 2 & 3 & -8 \\ 8 & -5 & 6 \end{vmatrix}$$

$$= 1\begin{vmatrix} 3 & -8 \\ -5 & 6 \end{vmatrix} - 0 + 6\begin{vmatrix} 2 & 3 \\ 8 & -5 \end{vmatrix}$$

$$= (18-40) + 6\,(-10-24) = -226$$

Thus the volume of the parallelepiped is $|-226| = 226$ cubic units.

19.

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} 2 & 3 & -2 \\ 1 & -1 & 0 \\ 2 & 0 & 3 \end{vmatrix}$$

$$= 2\begin{vmatrix} -1 & 0 \\ 0 & 3 \end{vmatrix} - 3\begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} + (-2)\begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix}$$

$$= -6 - 9 - 4 = -19$$

So the volume of the parallelepiped determined by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is $|-19| = 19$ cubic units.

20. $\mathbf{a} = \overrightarrow{PQ} = \langle 1,-1,2\rangle$, $\mathbf{b} = \overrightarrow{PR} = \langle 3,0,6\rangle$ and $\mathbf{c} = \overrightarrow{PS} = \langle 2,-2,-3\rangle$.

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 0 & 6 \\ 2 & -2 & -3 \end{vmatrix}$$

$$= 1\begin{vmatrix} 0 & 6 \\ -2 & -3 \end{vmatrix} - (-1)\begin{vmatrix} 3 & 6 \\ 2 & -3 \end{vmatrix} + 2\begin{vmatrix} 3 & 0 \\ 2 & -2 \end{vmatrix}$$

$$= 12 - 21 - 12 = -21$$

so the volume of the parallelepiped is 21 cubic units.