9.3 The Dot Product

A Click here for answers.

S Click here for solutions.

1–7 Find $\mathbf{a} \cdot \mathbf{b}$.

1. $\mathbf{a} = \langle 2, 5 \rangle, \quad \mathbf{b} = \langle -3, 1 \rangle$

2. $\mathbf{a} = \langle -2, -8 \rangle, \quad \mathbf{b} = \langle 6, -4 \rangle$

3. $\mathbf{a} = \langle 4, 7, -1 \rangle, \quad \mathbf{b} = \langle -2, 1, 4 \rangle$

4. $\mathbf{a} = \langle -1, -2, -3 \rangle, \quad \mathbf{b} = \langle 2, 8, -6 \rangle$

5. $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}, \quad \mathbf{b} = \mathbf{i} - 3\mathbf{j} + \mathbf{k}$

6. $\mathbf{a} = \mathbf{i} - \mathbf{k}, \quad \mathbf{b} = \mathbf{i} + 2\mathbf{j}$

7. $|\mathbf{a}| = 2, \quad |\mathbf{b}| = 3,$ the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\pi/3$

8–13 Find the angle between the vectors. (First find an exact expression and then approximate to the nearest degree.)

8. $\mathbf{a} = \langle 1, 2, 2 \rangle, \quad \mathbf{b} = \langle 3, 4, 0 \rangle$

9. $\mathbf{a} = \langle 6, 0, 2 \rangle, \quad \mathbf{b} = \langle 5, 3, -2 \rangle$

10. $\mathbf{a} = \langle 1, 2 \rangle, \quad \mathbf{b} = \langle 12, -5 \rangle$

11. $\mathbf{a} = \langle 3, 1 \rangle, \quad \mathbf{b} = \langle 2, 4 \rangle$

12. $\mathbf{a} = 6\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}, \quad \mathbf{b} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

13. $\mathbf{a} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}, \quad \mathbf{b} = 2\mathbf{j} - 3\mathbf{k}$

14–15 Find, correct to the nearest degree, the three angles of the triangle with the given vertices.

14. $A(1, 2, 3), \quad B(6, 1, 5), \quad C(-1, -2, 0)$

15. $P(0, -1, 6), \quad Q(2, 1, -3), \quad R(5, 4, 2)$

16–21 Determine whether the given vectors are orthogonal, parallel, or neither.

16. $\mathbf{a} = \langle 2, -4 \rangle, \quad \mathbf{b} = \langle -1, 2 \rangle$

17. $\mathbf{a} = \langle 2, -4 \rangle, \quad \mathbf{b} = \langle 4, 2 \rangle$

18. $\mathbf{a} = \langle 2, 8, -3 \rangle, \quad \mathbf{b} = \langle -1, 2, 5 \rangle$

19. $\mathbf{a} = \langle -1, 5, 2 \rangle, \quad \mathbf{b} = \langle 4, 2, -3 \rangle$

20. $\mathbf{a} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}, \quad \mathbf{b} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$

21. $\mathbf{a} = -\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}, \quad \mathbf{b} = 3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$

22. Find the values of x such that the vectors $\langle -3x, 2x \rangle$ and $\langle 4, x \rangle$ are orthogonal.

23. For what values of c is the angle between the vectors $\langle 1, 2, 1 \rangle$ and $\langle 1, 0, c \rangle$ equal to $60°$?

24–28 Find the direction cosines and direction angles of the vector. (Give the direction angles correct to the nearest degree.)

24. $\langle 1, 2, 2 \rangle$

25. $\langle -4, -1, 2 \rangle$

26. $-8\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$

27. $3\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}$

28. $\langle 2, 1.2, 0.8 \rangle$

29–30 Find the scalar and vector projections of $\mathbf{b}$ onto $\mathbf{a}$.

29. $\mathbf{a} = \langle 2, 3 \rangle, \quad \mathbf{b} = \langle 4, 1 \rangle$

30. $\mathbf{a} = \langle 3, -1 \rangle, \quad \mathbf{b} = \langle 2, 3 \rangle$

Answers

E Click here for exercises.

S Click here for solutions.

1. -1

2. 20

3. -5

4. 0

5. -11

6. 1

7. 3

8. $\cos^{-1}\left(\frac{11}{15}\right) \approx 43^\circ$

9. $\cos^{-1}\left(\frac{13}{2\sqrt{95}}\right) \approx 48^\circ$

10. $\cos^{-1}\left(\frac{2}{13\sqrt{5}}\right) \approx 86^\circ$

11. $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = 45^\circ$

12. $\cos^{-1}\left(\frac{1}{7\sqrt{3}}\right) \approx 85^\circ$

13. $\cos^{-1}\left(-\frac{4}{\sqrt{78}}\right) \approx 117^\circ$

14. $114^\circ, 33^\circ, 33^\circ$

15. $43^\circ, 58^\circ, 79^\circ$

16. Parallel

17. Orthogonal

18. Neither

19. Orthogonal

20. Orthogonal

21. Orthogonal

22. $0, 6$

23. $2 \pm \sqrt{3}$

24. $\frac{1}{3}, \frac{2}{3}, \frac{2}{3}; 71^\circ, 48^\circ, 48^\circ$

25. $-\frac{4}{\sqrt{21}}, -\frac{1}{\sqrt{21}}, \frac{2}{\sqrt{21}}; 151^\circ, 103^\circ, 64^\circ$

26. $-\frac{8}{\sqrt{77}}, \frac{3}{\sqrt{77}}, \frac{2}{\sqrt{77}}; 156^\circ, 70^\circ, 77^\circ$

27. $\frac{3}{5\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{4}{5\sqrt{2}}; 65^\circ, 45^\circ, 124^\circ$

28. $\frac{5}{\sqrt{38}}, \frac{3}{\sqrt{38}}, \frac{2}{\sqrt{38}}; 36^\circ, 61^\circ, 71^\circ$

29. $\frac{11}{\sqrt{13}}, \left\langle \frac{22}{13}, \frac{33}{13} \right\rangle$

30. $\frac{3}{\sqrt{10}}, \left\langle \frac{9}{10}, -\frac{3}{10} \right\rangle$

Solutions

E Click here for exercises.

A Click here for answers.

1. $\mathbf{a}\cdot\mathbf{b}=(2)(-3)+(5)(1)=-1$

2. $\mathbf{a}\cdot\mathbf{b}=(-2)(6)+(-8)(-4)=20$

3. $\mathbf{a}\cdot\mathbf{b}=(4)(-2)+(7)(1)+(-1)(4)=-5$

4. $\mathbf{a}\cdot\mathbf{b}=(-1)(2)+(-2)(8)+(-3)(-6)=0$

5. $\mathbf{a}\cdot\mathbf{b}=(2)(1)+(3)(-3)+(-4)(1)=-11$

6. $\mathbf{a}\cdot\mathbf{b}=(1)(1)+(0)(2)+(-1)(0)=1$

7. $\mathbf{a}\cdot\mathbf{b}=(2)(3)\cos\frac{\pi}{3}=6\cdot\frac{1}{2}=3$

8. $|\mathbf{a}|=\sqrt{1^2+2^2+2^2}=3$, $|\mathbf{b}|=\sqrt{3^2+4^2+0^2}=5$, $\mathbf{a}\cdot\mathbf{b}=3+8+0=11$, $\cos\theta=\frac{11}{3\cdot5}$, so $\theta=\cos^{-1}\left(\frac{11}{15}\right)\approx43^\circ$.

9. $|\mathbf{a}|=\sqrt{6^2+0^2+2^2}=2\sqrt{10}$, $|\mathbf{b}|=\sqrt{5^2+3^2+(-2)^2}=\sqrt{38}$, $\mathbf{a}\cdot\mathbf{b}=30+0+(-4)=26$, $\cos\theta=\frac{26}{2\sqrt{10}\sqrt{38}}$, so $\theta=\cos^{-1}\left(\frac{13}{2\sqrt{95}}\right)\approx48^\circ$.

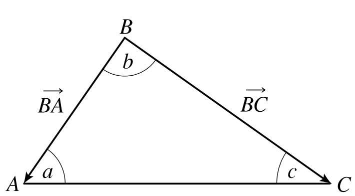
10. $|\mathbf{a}|=\sqrt{1^2+2^2}=\sqrt{5}$, $|\mathbf{b}|=\sqrt{12^2+(-5)^2}=\sqrt{13}$, $\mathbf{a}\cdot\mathbf{b}=12-10=2$, $\cos\theta=\frac{2}{13\sqrt{5}}$, so $\theta=\cos^{-1}\left(\frac{2}{13\sqrt{5}}\right)\approx86^\circ$.

11. $|\mathbf{a}|=\sqrt{3^2+1^2}=\sqrt{10}$, $|\mathbf{b}|=\sqrt{2^2+4^2}=\sqrt{5}$, $\mathbf{a}\cdot\mathbf{b}=6+4=10$, $\cos\theta=\frac{10}{\sqrt{10}\cdot2\sqrt{5}}=\frac{\sqrt{2}}{2}$ and $\theta=\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)=45^\circ$.

12. $|\mathbf{a}|=\sqrt{36+4+9}=7$, $|\mathbf{b}|=\sqrt{3}$, $\mathbf{a}\cdot\mathbf{b}=6-2-3=1$, $\cos\theta=\frac{1}{7\sqrt{3}}$, so $\theta=\cos^{-1}\left(\frac{1}{7\sqrt{3}}\right)\approx85^\circ$.

13. $|\mathbf{a}|=\sqrt{1+1+4}=\sqrt{6}$, $|\mathbf{b}|=\sqrt{4+9}=\sqrt{13}$, $\mathbf{a}\cdot\mathbf{b}=0+2-6=-4$, $\cos\theta=-\frac{4}{\sqrt{78}}$, so $\theta=\cos^{-1}\left(-\frac{4}{\sqrt{78}}\right)\approx117^\circ$.

14.



Let a, b and c be the angles at vertices A, B and C respectively. Then a is the angle between vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$, b is the angle between vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$, and c is the angle between vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$.

Thus

$$\cos a=\frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|}=\frac{1}{\sqrt{30\cdot29}}\langle5,-1,2\rangle\cdot\langle-2,-4,-3\rangle=\frac{1}{\sqrt{870}}(-10+4-6)=-\frac{12}{\sqrt{870}}$$

and $a=\cos^{-1}\left(-\frac{12}{\sqrt{870}}\right)\approx114^\circ$. Similarly

$$\cos b=\frac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|}=\frac{1}{\sqrt{30\cdot83}}\langle5,-1,2\rangle\cdot\langle-7,-3,-5\rangle=\frac{1}{\sqrt{2490}}(35-3+10)=\frac{42}{\sqrt{2490}}$$

so $b=\cos^{-1}\left(\frac{42}{\sqrt{2490}}\right)\approx33^\circ$, and

$$\cos c=\frac{\overrightarrow{CA}\cdot\overrightarrow{CB}}{|\overrightarrow{CA}||\overrightarrow{CB}|}=\frac{1}{\sqrt{29\cdot83}}\langle2,4,3\rangle\cdot\langle7,3,5\rangle=\frac{1}{\sqrt{2407}}(14+12+15)=\frac{41}{\sqrt{2407}}$$

so $c=\cos^{-1}\left(\frac{41}{\sqrt{2407}}\right)\approx33^\circ$.

Alternate Solution: Apply the Law of Cosines three times as follows: $\cos a=\frac{|\overrightarrow{BC}|^2-|\overrightarrow{AB}|^2-|\overrightarrow{AC}|^2}{2|\overrightarrow{AB}||\overrightarrow{AC}|}$,

$\cos b=\frac{|\overrightarrow{AC}|^2-|\overrightarrow{AB}|^2-|\overrightarrow{BC}|^2}{2|\overrightarrow{AB}||\overrightarrow{BC}|}$, and

$\cos c=\frac{|\overrightarrow{AB}|^2-|\overrightarrow{AC}|^2-|\overrightarrow{BC}|^2}{2|\overrightarrow{AC}||\overrightarrow{BC}|}$.

15. As in Problem 14, let p, q and r be the angles at vertices P, Q and R. Then p is the angle between vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, q is the angle between vectors $\overrightarrow{QP}$ and $\overrightarrow{QR}$, and r is the angle between vectors $\overrightarrow{RP}$ and $\overrightarrow{RQ}$. Thus

$\cos p=\frac{\overrightarrow{PQ}\cdot\overrightarrow{PR}}{|\overrightarrow{PQ}||\overrightarrow{PR}|}=\frac{\langle2,2,-9\rangle\cdot\langle5,5,-4\rangle}{\sqrt{89}\sqrt{66}}=\frac{56}{\sqrt{5874}}$, so

$p=\cos^{-1}\left(\frac{56}{\sqrt{5874}}\right)\approx43^\circ$;

$\cos q=\frac{\overrightarrow{QP}\cdot\overrightarrow{QR}}{|\overrightarrow{QP}||\overrightarrow{QR}|}=\frac{\langle-2,-2,9\rangle\cdot\langle3,3,5\rangle}{\sqrt{89}\sqrt{43}}=\frac{33}{\sqrt{3827}}$,

so $q=\cos^{-1}\left(\frac{33}{\sqrt{3827}}\right)\approx58^\circ$; and

$\cos r=\frac{\overrightarrow{RP}\cdot\overrightarrow{RQ}}{|\overrightarrow{RP}||\overrightarrow{RQ}|}=\frac{\langle-5,-5,4\rangle\cdot\langle-3,-3,-5\rangle}{\sqrt{66}\sqrt{43}}=\frac{10}{\sqrt{2838}}$, so

$r=\cos^{-1}\left(\frac{10}{\sqrt{2838}}\right)\approx79^\circ$.

Alternate Solution: Apply the Law of Cosines three times as follows: $\cos p=\frac{|\overrightarrow{QR}|^2-|\overrightarrow{PQ}|^2-|\overrightarrow{PR}|^2}{2|\overrightarrow{PQ}||\overrightarrow{PR}|}$,

$\cos q=\frac{|\overrightarrow{PR}|^2-|\overrightarrow{PQ}|^2-|\overrightarrow{QR}|^2}{2|\overrightarrow{PQ}||\overrightarrow{QR}|}$, and

$\cos r=\frac{|\overrightarrow{PQ}|^2-|\overrightarrow{PR}|^2-|\overrightarrow{QR}|^2}{2|\overrightarrow{PR}||\overrightarrow{QR}|}$.

16. Since $\mathbf{a}=-2\mathbf{b}$, $\mathbf{a}$ and $\mathbf{b}$ are parallel vectors (and thus not orthogonal).

17. $\mathbf{a}\cdot\mathbf{b}=8+(-8)=0$, so $\mathbf{a}$ and $\mathbf{b}$ are orthogonal (and not parallel).

18. $\mathbf{a}\cdot\mathbf{b} = -2 + 16 + (-15) \neq 0$, so $\mathbf{a}$ and $\mathbf{b}$ are not orthogonal. Also since $\mathbf{a}$ is not a scalar multiple of $\mathbf{b}$, $\mathbf{a}$ and $\mathbf{b}$ are not parallel.

19. $\mathbf{a}\cdot\mathbf{b} = -4 + 10 + (-6) = 0$, so $\mathbf{a}$ and $\mathbf{b}$ are orthogonal (and not parallel).

20. $\mathbf{a}\cdot\mathbf{b} = 3 + (-1) + (-2) = 0$, so $\mathbf{a}$ and $\mathbf{b}$ are orthogonal.

21. $\mathbf{a}\cdot\mathbf{b} = (-1)(3) + (2)(4) + (5)(-1) = 0$, so $\mathbf{a}$ and $\mathbf{b}$ are orthogonal (and not parallel).

22. For the two vectors to be orthogonal, we need $\langle -3x, 2x\rangle \cdot \langle 4, x\rangle = 0 \quad\Leftrightarrow\quad (-3x)(4) + (2x)(x) = 0$ $\Leftrightarrow\quad -12x + 2x^2 = 0 \quad\Leftrightarrow\quad 2x(x-6) = 0 \quad\Leftrightarrow\quad x = 0$ or $x = 6$.

23. Using Theorem 3, we need $\langle 1,2,1\rangle \cdot \langle 1,0,c\rangle = |\langle 1,2,1\rangle|\,|\langle 1,0,c\rangle| \cos 60^\circ \quad\Leftrightarrow$ $1 + c = \sqrt{6}\sqrt{1+c^2}\cdot\frac{1}{2} \quad\Leftrightarrow\quad 2(1+c) = \sqrt{6}\sqrt{1+c^2}$. Squaring both sides gives $6(1+c^2) = 4(1+2c+c^2)$. Thus $6 + 6c^2 = 4 + 8c + 4c^2$ or $2c^2 - 8c + 2 = 0$ and $c = \frac{4\pm\sqrt{16-4}}{2} = 2 \pm \sqrt{3}$. Each of these values for c can be checked to show it gives a solution.

24. Since $|\langle 1,2,2\rangle| = \sqrt{1+4+4} = 3$, using (8) and (9) we have $\cos\alpha = \frac{1}{3}$, $\cos\beta = \frac{2}{3}$ and $\cos\gamma = \frac{2}{3}$, while $\alpha = \cos^{-1}\left(\frac{1}{3}\right) \approx 71^\circ$ and $\beta = \gamma = \cos^{-1}\left(\frac{2}{3}\right) \approx 48^\circ$.

25. $|\langle -4,-1,2\rangle| = \sqrt{16+1+4} = \sqrt{21}$, so $\cos\alpha = -\frac{4}{\sqrt{21}}$, $\cos\beta = -\frac{1}{\sqrt{21}}$ and $\cos\gamma = \frac{2}{\sqrt{21}}$, while $\alpha = \cos^{-1}\left(\frac{-4}{\sqrt{21}}\right) \approx 151^\circ$, $\beta = \cos^{-1}\left(\frac{-1}{\sqrt{21}}\right) \approx 103^\circ$ and $\gamma = \cos^{-1}\left(\frac{2}{\sqrt{21}}\right) \approx 64^\circ$.

26. $|-8\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}| = \sqrt{64+9+4} = \sqrt{77}$, so $\cos\alpha = -\frac{8}{\sqrt{77}}$, $\cos\beta = \frac{3}{\sqrt{77}}$ and $\cos\gamma = \frac{2}{\sqrt{77}}$, while $\alpha = \cos^{-1}\left(\frac{-8}{\sqrt{77}}\right) \approx 156^\circ$, $\beta = \cos^{-1}\left(\frac{3}{\sqrt{77}}\right) \approx 70^\circ$ and $\gamma = \cos^{-1}\left(\frac{2}{\sqrt{77}}\right) \approx 77^\circ$.

27. $|3\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}| = \sqrt{9+25+16} = 5\sqrt{2}$, so $\cos\alpha = \frac{3}{5\sqrt{2}}$, $\cos\beta = \frac{1}{\sqrt{2}}$ and $\cos\gamma = -\frac{4}{5\sqrt{2}}$, while $\alpha = \cos^{-1}\left(\frac{3}{5\sqrt{2}}\right) \approx 65^\circ$, $\beta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45^\circ$ and $\gamma = \cos^{-1}\left(\frac{-4}{5\sqrt{2}}\right) \approx 124^\circ$.

28. $|\langle 2, 1.2, 0.8\rangle| = \sqrt{4 + 1.44 + 0.64} = \frac{5}{5}\sqrt{6.08} = \frac{\sqrt{152}}{5}$, so $\cos\alpha = \frac{10}{\sqrt{152}} = \frac{5}{\sqrt{38}}$, $\cos\beta = \frac{6}{\sqrt{152}} = \frac{3}{\sqrt{38}}$ and $\cos\gamma = \frac{4}{\sqrt{152}} = \frac{2}{\sqrt{38}}$, while $\alpha = \cos^{-1}\left(\frac{5}{\sqrt{38}}\right) \approx 36^\circ$, $\beta = \cos^{-1}\left(\frac{3}{\sqrt{38}}\right) \approx 61^\circ$ and $\gamma = \cos^{-1}\left(\frac{2}{\sqrt{38}}\right) \approx 71^\circ$.

29. $|\mathbf{a}| = \sqrt{4+9} = \sqrt{13}$. The scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$$\operatorname{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \frac{2\cdot 4 + 3\cdot 1}{\sqrt{13}} = \frac{11}{\sqrt{13}}.$$

The vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$$\operatorname{proj}_{\mathbf{a}}\mathbf{b} = \frac{11}{\sqrt{13}}\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{11}{\sqrt{13}}\cdot\frac{1}{\sqrt{13}}\langle 2,3\rangle = \frac{11}{13}\langle 2,3\rangle = \left\langle \frac{22}{13}, \frac{33}{13}\right\rangle$$

30. $|\mathbf{a}| = \sqrt{9+1} = \sqrt{10}$. The scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$$\operatorname{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \frac{3\cdot 2 - 1\cdot 3}{\sqrt{10}} = \frac{3}{\sqrt{10}}.$$

The vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$$\operatorname{proj}_{\mathbf{a}}\mathbf{b} = \frac{3}{\sqrt{10}}\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{3}{\sqrt{10}}\cdot\frac{1}{\sqrt{10}}\langle 3,-1\rangle = \frac{3}{10}\langle 3,-1\rangle = \left\langle \frac{9}{10}, -\frac{3}{10}\right\rangle$$