8.8 The Binomial Series

A Click here for answers.

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1–6 Use the binomial series to expand the function as a power series. State the radius of convergence.

1. $\sqrt[3]{1 + x^2}$

2. $\dfrac{x}{\sqrt{1 - x}}$

3. $\dfrac{1}{\sqrt{2 + x}}$

4. $\dfrac{x^2}{\sqrt{1 - x^3}}$

5. $\left(\dfrac{x}{1 - x}\right)^5$

6. $\sqrt[5]{x - 1}$

7. (a) Expand $1/\sqrt{1 + x}$ as a power series.
(b) Use part (a) to estimate $1/\sqrt{1.1}$ correct to three decimal places.

8. (a) Expand $\sqrt[3]{8 + x}$ as a power series.
(b) Use part (a) to estimate $\sqrt[3]{8.2}$ correct to four decimal places.

Answers

E Click here for exercises.

S Click here for solutions.

1. $1+\dfrac{x^2}{3}+\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^{n-1}\cdot 2\cdot 5\cdot 8\cdot\cdots\cdot(3n-4)\,x^{2n}}{3^n\,n!}$, $R=1$

2. $x+\displaystyle\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2^n n!}x^{n+1}$, $R=1$

3. $\dfrac{\sqrt{2}}{2}\left[1+\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n\cdot 1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)\,x^n}{2^{2n}\cdot n!}\right]$, $R=2$

4. $x^2+\displaystyle\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)\,x^{3n+2}}{2^n\cdot n!}$, $R=1$

5. $\displaystyle\sum_{n=0}^{\infty}\frac{(n+4)!}{4!\cdot n!}x^{n+5}$, $R=1$

6. $-1+\dfrac{x}{5}+\displaystyle\sum_{n=2}^{\infty}\frac{4\cdot 9\cdots(5n-6)\,x^n}{5^n\cdot n!}$, $R=1$

7. (a) $1+\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n\,1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2^n\cdot n!}x^n$

(b) 0.953

8. (a) $2\left[1+\dfrac{x}{24}+\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^{n-1}\cdot 2\cdot 5\cdot\cdots\cdot(3n-4)\,x^n}{24^n\cdot n!}\right]$

(b) 2.0165

Solutions

E Click here for exercises.

A Click here for answers.

1. $(1+x^2)^{1/3} = \sum_{n=0}^{\infty} \binom{\frac{1}{3}}{n} x^{2n}$

$$= 1 + \frac{x^2}{3} + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!} x^4 + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!} x^6 + \cdots$$

$$= 1 + \frac{x^2}{3} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} \cdot 2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n-4)\, x^{2n}}{3^n\, n!}$$

with $R = 1$.

2. $[1+(-x)]^{-1/2} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-x)^n$

$$= 1 + \left(-\tfrac{1}{2}\right)(-x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!} (-x)^2 + \cdots$$

$$= 1 + \frac{x}{2} + \frac{1 \cdot 3}{2^2 2!} x^2 + \frac{1 \cdot 3 \cdot 5}{2^3 3!} x^3 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2^4 4!} x^4 + \cdots$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2^n n!} x^n$$

so $\dfrac{x}{\sqrt{1-x}} = x + \sum_{n=1}^{\infty} \dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2^n n!} x^{n+1}$ with $R = 1$.

3. $(2+x)^{-1/2} = \dfrac{1}{\sqrt{2}} \left(1 + \dfrac{x}{2}\right)^{-1/2}$

$$= \frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \left(\frac{x}{2}\right)^n$$

$$= \frac{\sqrt{2}}{2} \left[1 + \left(-\tfrac{1}{2}\right)\left(\frac{x}{2}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!} \left(\frac{x}{2}\right)^2 + \cdots\right]$$

$$= \frac{\sqrt{2}}{2} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)\, x^n}{2^{2n} \cdot n!}\right]$$

with $|x/2| < 1$ so $|x| < 2$ and $R = 2$.

4. $\left[1 + \left(-x^3\right)\right]^{-1/2} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \left(-x^3\right)^n$

$$= 1 + \left(-\tfrac{1}{2}\right)\left(-x^3\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!} \left(-x^3\right)^2 + \cdots$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)\, x^{3n}}{2^n \cdot n!}$$

so $\dfrac{x^2}{\sqrt{1-x^3}} = x^2 + \sum_{n=1}^{\infty} \dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)\, x^{3n+2}}{2^n \cdot n!}$

with $R = 1$.

5. $(1-x)^{-5} = 1 + (-5)(-x) + \dfrac{(-5)(-6)}{2!}(-x)^2 + \dfrac{(-5)(-6)(-7)}{3!}(-x)^3 + \cdots$

$$= 1 + \sum_{n=1}^{\infty} \frac{5 \cdot 6 \cdot 7 \cdots (n+4)}{n!} x^n = \sum_{n=0}^{\infty} \frac{(n+4)!}{4! \cdot n!} x^n$$

$\Rightarrow \dfrac{x^5}{(1-x)^5} = \sum_{n=0}^{\infty} \dfrac{(n+4)!}{4! \cdot n!} x^{n+5}$ or

$\sum_{n=0}^{\infty} \dfrac{(n+1)(n+2)(n+3)(n+4)}{24} x^{n+5}$, with $R = 1$.

6. $\sqrt[5]{x-1} = -[1+(-x)]^{1/5} = -\sum_{n=0}^{\infty} \binom{\frac{1}{5}}{n} (-x)^n$

$$= -\left[1 + \tfrac{1}{5}(-x) + \frac{\left(\frac{1}{5}\right)\left(-\frac{4}{5}\right)}{2!}(-x)^2 + \frac{\left(\frac{1}{5}\right)\left(-\frac{4}{5}\right)\left(-\frac{9}{5}\right)}{3!}(-x)^3 + \cdots\right]$$

$$= -1 + \frac{x}{5} + \sum_{n=2}^{\infty} \frac{4 \cdot 9 \cdots (5n-6)\, x^n}{5^n \cdot n!} \text{ with } R = 1.$$

7. (a) $(1+x)^{-1/2} = 1 + \left(-\tfrac{1}{2}\right)x + \dfrac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}x^2 + \dfrac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}x^3 + \cdots$

$$= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n\, 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2^n \cdot n!} x^n$$

(b) Take $x = 0.1$ in the above series.

$\frac{1 \cdot 3 \cdot 5 \cdot 7}{2^4 4!}(0.1)^4 < 0.00003$, so

$\frac{1}{\sqrt{1.1}} \approx 1 - \frac{0.1}{2} + \frac{1 \cdot 3}{2^2 \cdot 2!}(0.1)^2 - \frac{1 \cdot 3 \cdot 5}{2^3 \cdot 3!}(0.1)^3 \approx 0.953$

8. (a) $(8+x)^{1/3} = 2\left(1 + \dfrac{x}{8}\right)^{1/3} = 2\sum_{n=0}^{\infty} \binom{\frac{1}{3}}{n} \left(\dfrac{x}{8}\right)^n$

$$= 2\left[1 + \frac{1}{3}\left(\frac{x}{8}\right) + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}\left(\frac{x}{8}\right)^2 + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}\left(\frac{x}{8}\right)^3 + \cdots\right]$$

$$= 2\left[1 + \frac{x}{24} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} \cdot 2 \cdot 5 \cdot \cdots \cdot (3n-4)\, x^n}{24^n \cdot n!}\right]$$

(b) $(8+0.2)^{1/3} = 2\left[1 + \frac{0.2}{24} - \frac{(0.2)^2}{24^2} + \frac{2 \cdot 5 (0.2)^3}{24^3 \cdot 3!} - \cdots\right]$

$$\approx 2\left[1 + \tfrac{0.2}{24} - \tfrac{(0.2)^2}{24^2}\right]$$

since $2 \cdot \frac{2 \cdot 5 (0.2)^3}{24^3 \cdot 3!} \approx 0.000002$, so $\sqrt[3]{8.2} \approx 2.0165$.