

8.7 Taylor and Maclaurin Series

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1–2 ■ Find the Maclaurin series for $f(x)$ using the definition of a Maclaurin series. [Assume that f has a power series expansion. Do not show that $R_n(x) \rightarrow 0$.] Also find the associated radius of convergence.

1. $f(x) = \frac{1}{(1+x)^2}$ 2. $f(x) = \frac{x}{1-x}$

3–6 ■ Find the Taylor series for $f(x)$ centered at the given value of a . [Assume that f has a power series expansion. Do not show that $R_n(x) \rightarrow 0$.]

3. $f(x) = 1/x$, $a = 1$ 4. $f(x) = \sqrt{x}$, $a = 4$
 5. $f(x) = \sin x$, $a = \pi/4$ 6. $f(x) = \cos x$, $a = -\pi/4$

7–13 ■ Use a Maclaurin series derived in this section to obtain the Maclaurin series for the given function.

7. $f(x) = e^{3x}$ 8. $f(x) = \sin 2x$
 9. $f(x) = x^2 \cos x$ 10. $f(x) = \cos(x^3)$
 11. $f(x) = x \sin(x/2)$ 12. $f(x) = xe^{-x}$

13. $f(x) = \begin{cases} \frac{1 - \cos x}{x^2} & \text{if } x \neq 0 \\ \frac{1}{2} & \text{if } x = 0 \end{cases}$

14–15 ■ Find the Maclaurin series of f (by any method) and its radius of convergence. Graph f and its first few Taylor polynomials on the same screen. What do you notice about the relationship between these polynomials and f ?

14. $f(x) = 1/\sqrt{1+2x}$ 15. $f(x) = (1+x)^{-3}$

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16. Find the Maclaurin series for $\ln(1+x)$ and use it to calculate $\ln 1.1$ correct to five decimal places.

17–18 ■ Evaluate the indefinite integral as an infinite series.

17. $\int \sin(x^2) dx$ 18. $\int e^{x^3} dx$

19–20 ■ Use series to approximate the definite integral correct to three decimal places.

19. $\int_0^1 \sin(x^2) dx$ 20. $\int_0^{0.5} \cos(x^2) dx$

21. Use multiplication or division of power series to find the first three nonzero terms in the Maclaurin series for

$$y = \frac{\ln(1-x)}{e^x}$$

22–24 ■ Find the sum of the series.

22. $\sum_{n=2}^{\infty} \frac{x^{3n+1}}{n!}$ 23. $\sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!}$

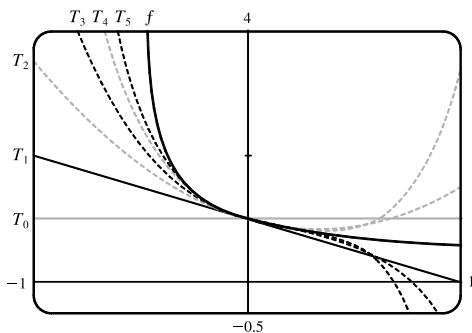
24. $\sum_{n=0}^{\infty} \frac{x^n}{2^n(n+1)!}$

25. Show that $e^x > 1+x$ for all $x > 0$.

Answers

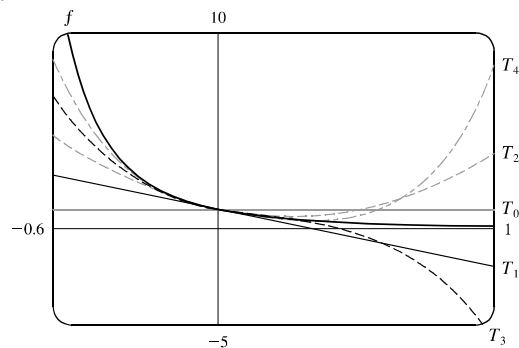
E Click here for exercises.

1. $\sum_{n=0}^{\infty} (-1)^n (n+1) x^n, R = 1$
2. $\sum_{n=1}^{\infty} x^n, R = 1$
3. $\sum_{n=0}^{\infty} (-1)^n (x-1)^n, R = 1$
4. $2 + \frac{x-4}{4} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^{3n-1} n!} (x-4)^n,$
 $R = 4$
5. $\frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} (-1)^n \left[\frac{1}{(2n)!} \left(x - \frac{\pi}{4}\right)^{2n} + \frac{1}{(2n+1)!} \left(x - \frac{\pi}{4}\right)^{2n+1} \right], R = \infty$
6. $\frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} \frac{(-1)^{n(n-1)/2} \left(x + \frac{\pi}{4}\right)^n}{n!}, R = \infty$
7. $\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}, R = \infty$
8. $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{(2n+1)!}, R = \infty$
9. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n)!}, R = \infty$
10. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!}, R = \infty$
11. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+1)! 2^{2n+1}}, R = \infty$
12. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{(n-1)!}, R = \infty$
13. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+2)!}, R = \infty$
14. $\sum_{n=0}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!} x^n, R = \frac{1}{2}.$



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$$15. \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)(n+2)x^n}{2}, R = 1$$



16. 0.09531
17. $C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(4n+3)(2n+1)!}$
18. $C + \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)n!}$
19. 0.310
20. 0.497
21. $-x + \frac{x^2}{2} - \frac{x^3}{3} + \dots$
22. $x(e^{x^3} - 1 - x^3)$
23. $e^x - 1$
24. $\frac{2}{x}(e^{x/2} - 1)$

Solutions

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1.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1+x)^{-2}$	1
1	$-2(1+x)^{-3}$	-2
2	$2 \cdot 3(1+x)^{-4}$	$2 \cdot 3$
3	$-2 \cdot 3 \cdot 4(1+x)^{-5}$	$-2 \cdot 3 \cdot 4$
4	$2 \cdot 3 \cdot 4 \cdot 5(1+x)^{-6}$	$2 \cdot 3 \cdot 4 \cdot 5$
...

So $f^{(n)}(0) = (-1)^n (n+1)!$ and

$$\begin{aligned} \frac{1}{(1+x)^2} &= \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)!}{n!} x^n \\ &= \sum_{n=0}^{\infty} (-1)^n (n+1) x^n \end{aligned}$$

If $a_n = (-1)^n (n+1) x^n$, then $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = |x|$, so

$R = 1$.

2.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$x/(1-x)$	0
1	$(1-x)^{-2}$	1
2	$2(1-x)^{-3}$	2
3	$3 \cdot 2(1-x)^{-4}$	$3 \cdot 2$
4	$4 \cdot 3 \cdot 2(1-x)^{-5}$	$4 \cdot 3 \cdot 2$
...

$f^{(n)}(0) = n!$ except when $n = 0$, so

$$\frac{x}{1-x} = \sum_{n=1}^{\infty} \frac{n!}{n!} x^n = \sum_{n=1}^{\infty} x^n. \quad \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = |x| < 1 \text{ for convergence, so } R = 1.$$

3.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	x^{-1}	1
1	$-x^{-2}$	-1
2	$2x^{-3}$	2
3	$-3 \cdot 2x^{-4}$	$-3 \cdot 2$
4	$4 \cdot 3 \cdot 2x^{-5}$	$4 \cdot 3 \cdot 2$
...

So $f^{(n)}(1) = (-1)^n n!$, and

$$\frac{1}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{n!} (x-1)^n = \sum_{n=0}^{\infty} (-1)^n (x-1)^n. \text{ If}$$

$a_n = (-1)^n (x-1)^n$ then $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = |x-1| < 1$ for convergence, so $0 < x < 2$ and $R = 1$.

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4.

n	$f^{(n)}(x)$	$f^{(n)}(4)$
0	$x^{1/2}$	2
1	$\frac{1}{2}x^{-1/2}$	2^{-2}
2	$-\frac{1}{4}x^{-3/2}$	-2^{-5}
3	$\frac{3}{8}x^{-5/2}$	$3 \cdot 2^{-8}$
4	$-\frac{15}{16}x^{-7/2}$	$-15 \cdot 2^{-11}$
...

$$f^{(n)}(4) = \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^{3n-1}} \text{ for } n \geq 2, \text{ so}$$

$$\sqrt{x} = 2 + \frac{x-4}{4} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^{3n-1} n!} (x-4)^n$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{|x-4|}{8} \lim_{n \rightarrow \infty} \left(\frac{2n-1}{n+1} \right) = \frac{|x-4|}{4} < 1$$

for convergence, so $|x-4| < 4 \Rightarrow R = 4$.

5.

n	$f^{(n)}(x)$	$f^{(n)}(\frac{\pi}{4})$
0	$\sin x$	$\sqrt{2}/2$
1	$\cos x$	$\sqrt{2}/2$
2	$-\sin x$	$-\sqrt{2}/2$
3	$-\cos x$	$-\sqrt{2}/2$
4	$\sin x$	$\sqrt{2}/2$
...

$$\begin{aligned} \sin x &= f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)(x - \frac{\pi}{4}) + \frac{f''\left(\frac{\pi}{4}\right)}{2!}(x - \frac{\pi}{4})^2 \\ &\quad + \frac{f^{(3)}\left(\frac{\pi}{4}\right)}{3!}(x - \frac{\pi}{4})^3 + \frac{f^{(4)}\left(\frac{\pi}{4}\right)}{4!}(x - \frac{\pi}{4})^4 + \cdots \\ &= \frac{\sqrt{2}}{2} \left[1 + (x - \frac{\pi}{4}) - \frac{1}{2!}(x - \frac{\pi}{4})^2 \right. \\ &\quad \left. - \frac{1}{3!}(x - \frac{\pi}{4})^3 + \frac{1}{4!}(x - \frac{\pi}{4})^4 + \cdots \right] \\ &= \frac{\sqrt{2}}{2} \left[1 - \frac{1}{2!}(x - \frac{\pi}{4})^2 + \frac{1}{4!}(x - \frac{\pi}{4})^4 - \cdots \right] \\ &\quad + \frac{\sqrt{2}}{2} \left[(x - \frac{\pi}{4}) - \frac{1}{3!}(x - \frac{\pi}{4})^3 + \cdots \right] \\ &= \frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} (-1)^n \left[\frac{1}{(2n)!} (x - \frac{\pi}{4})^{2n} \right. \\ &\quad \left. + \frac{1}{(2n+1)!} (x - \frac{\pi}{4})^{2n+1} \right] \end{aligned}$$

The series can also be written in the more elegant form

$$\sin x = \frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} \frac{(-1)^{n(n-1)/2} (x - \frac{\pi}{4})^n}{n!}. \text{ If}$$

$$a_n = \frac{(-1)^{n(n-1)/2} (x - \frac{\pi}{4})^n}{n!}, \text{ then}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{|x - \frac{\pi}{4}|}{n+1} = 0 < 1 \text{ for all } x, \text{ so}$$

$R = \infty$.

6.

n	$f^{(n)}(x)$	$f^{(n)}(-\frac{\pi}{4})$
0	$\cos x$	$\frac{\sqrt{2}}{2}$
1	$-\sin x$	$\frac{\sqrt{2}}{2}$
2	$-\cos x$	$-\frac{\sqrt{2}}{2}$
3	$\sin x$	$-\frac{\sqrt{2}}{2}$
4	$\cos x$	$\frac{\sqrt{2}}{2}$
...

$f^{(n)}(-\frac{\pi}{4}) = (-1)^{n(n-1)/2} \frac{\sqrt{2}}{2}$, so

$$\begin{aligned} \cos x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(-\frac{\pi}{4})}{n!} (x + \frac{\pi}{4})^n \\ &= \frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} \frac{(-1)^{n(n-1)/2} (x + \frac{\pi}{4})^n}{n!} \end{aligned}$$

with $R = \infty$ by the Ratio Test (as in Problem 5).

7. $e^{3x} = \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} = \sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$, with $R = \infty$.

8. $\sin 2x = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{(2n+1)!}$,
 $R = \infty$

9. $x^2 \cos x = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n)!}$,
 $R = \infty$

10. $\cos(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!}$, $R = \infty$.

11. $x \sin(\frac{x}{2}) = x \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+1}}{(2n+1)!}$
 $= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+1)! 2^{2n+1}}$ with $R = \infty$.

12. $x e^{-x} = x \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n!}$
 $= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{(n-1)!}$, $R = \infty$.

13. $\frac{1 - \cos x}{x^2} = x^{-2} \left[1 - \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \right]$
 $= x^{-2} \left[- \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \right]$
 $= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-2}}{(2n)!}$
 $= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+2)!}$

since the series is equal to $\frac{1}{2}$ when $x = 0$; $R = \infty$.

14.

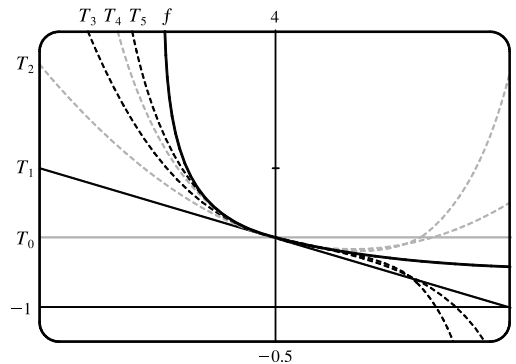
n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1+2x)^{-1/2}$	1
1	$-\frac{1}{2}(1+2x)^{-3/2}(2)$	-1
2	$\frac{3}{2}(1+2x)^{-5/2}(2)$	3
3	$-3 \cdot \frac{5}{2}(1+2x)^{-7/2}(2)$	$-3 \cdot 5$
...

$f^{(n)}(0) = (-1)^n 1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1)$, so

$$\begin{aligned} (1+2x)^{-1/2} &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!} x^n \end{aligned}$$

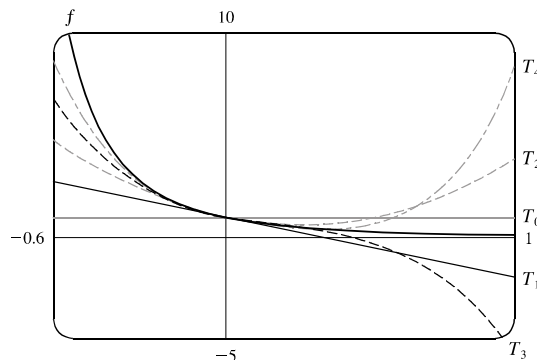
$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{2n+1}{n+1} |x| = 2|x| < 1$ for convergence, so $R = \frac{1}{2}$.

Another method: Use Exercise 33 from the text and differentiate.



15. $f(x) = (1+x)^{-3} = -\frac{1}{2} \frac{d}{dx} \left[\frac{1}{(1+x)^2} \right]$
 $= -\frac{1}{2} \frac{d}{dx} \left[\sum_{n=0}^{\infty} (-1)^n (n+1) x^n \right]$ [from Problem 1]
 $= -\frac{1}{2} \sum_{n=1}^{\infty} (-1)^n n (n+1) x^{n-1}$
 $= \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)(n+2) x^n}{2}$

with $R = 1$ since that is the R in Problem 1.



$$16. \ln(1+x) = \int \frac{dx}{1+x} = \int \sum_{n=0}^{\infty} (-1)^n x^n dx$$

$$= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$$

with $C = 0$ and $R = 1$, so $\ln(1.1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (0.1)^n}{n}$.

This is an alternating series with

$$b_5 = \frac{(0.1)^5}{5} = 0.000002, \text{ so to five decimal places,}$$

$$\ln(1.1) \approx \sum_{n=1}^4 \frac{(-1)^{n-1} (0.1)^n}{n} \approx 0.09531.$$

$$17. \int \sin(x^2) dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!} dx$$

$$= \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!} dx$$

$$= C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(4n+3)(2n+1)!}$$

$$18. \int e^{x^3} dx = \int \sum_{n=0}^{\infty} \frac{(x^3)^n}{n!} dx = C + \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)n!} \text{ with}$$

$$R = \infty.$$

19. Using our series from Problem 17, we get

$$\int_0^1 \sin(x^2) dx = \sum_{n=0}^{\infty} \left[\frac{(-1)^n x^{4n+3}}{(4n+3)(2n+1)!} \right]_0^1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+3)(2n+1)!}$$

$$\text{and } |c_3| = \frac{1}{75,600} < 0.000014, \text{ so by the}$$

Alternating Series Estimation Theorem, we have

$$\sum_{n=0}^2 \frac{(-1)^n}{(4n+3)(2n+1)!} = \frac{1}{3} - \frac{1}{42} + \frac{1}{1320} \approx 0.310$$

(correct to three decimal places).

$$20. \cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n)!}, \text{ so}$$

$$\int_0^{0.5} \cos(x^2) dx = \int_0^{0.5} \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!} dx$$

$$= \sum_{n=0}^{\infty} \left[\frac{(-1)^n x^{4n+1}}{(4n+1)(2n)!} \right]_0^{0.5}$$

$$= 0.5 - \frac{(0.5)^5}{5 \cdot 2!} + \frac{(0.5)^9}{9 \cdot 4!} - \dots$$

$$\text{but } \frac{(0.5)^9}{9 \cdot 4!} \approx 0.000009, \text{ so by the}$$

Alternating Series Estimation Theorem,

$$\int_0^{0.5} \cos(x^2) dx \approx 0.5 - \frac{(0.5)^5}{5 \cdot 2!} \approx 0.497 \text{ (correct to three decimal places).}$$

$$21. \begin{array}{r} -x + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \dots \\ 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots \\ \hline -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \dots \\ -x - x^2 - \frac{1}{2}x^3 - \dots \\ \hline \frac{1}{2}x^2 + \frac{1}{6}x^3 - \dots \\ \frac{1}{2}x^2 + \frac{1}{2}x^3 + \dots \\ \hline -\frac{1}{3}x^3 + \dots \\ -\frac{1}{3}x^3 + \dots \\ \hline \dots \end{array}$$

From Example 6 in Section 11.9, we have

$\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \dots$, $|x| < 1$. Therefore,

$$y = \frac{\ln(1-x)}{e^x} = \frac{-x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \dots}{1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots}.$$

So by the long division above, $\frac{\ln(1-x)}{e^x} = -x + \frac{x^2}{2} - \frac{x^3}{3} + \dots$, $|x| < 1$.

$$22. \sum_{n=2}^{\infty} \frac{x^{3n+1}}{n!} = x \sum_{n=2}^{\infty} \frac{(x^3)^n}{n!} = x \left[\sum_{n=0}^{\infty} \frac{(x^3)^n}{n!} - 1 - x^3 \right]$$

$$= x(e^{x^3} - 1 - x^3) \text{ by (11)}$$

$$23. \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} = \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$= \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right) - 1$$

$$= e^x - 1 \text{ by (11)}$$

$$24. \sum_{n=0}^{\infty} \frac{x^n}{2^n(n+1)!} = \sum_{n=0}^{\infty} \frac{(x/2)^n}{(n+1)!} = \frac{2}{x} \sum_{n=0}^{\infty} \frac{(x/2)^n}{(n+1)!}$$

$$= \frac{2}{x} \left[(x/2) + \frac{(x/2)^2}{2!} + \frac{(x/2)^3}{3!} + \dots \right]$$

$$= \frac{2}{x} (e^{x/2} - 1)$$

25. By (11), $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$, but for $x > 0$, all of the terms after the first two on the RHS are positive, so $e^x > 1 + x$ for $x > 0$.