

8.6 Representations of Functions as Power Series

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1–7 Find a power series representation for the function and determine the interval of convergence.

1. $f(x) = \frac{x}{1-x}$

2. $f(x) = \frac{1}{4+x^2}$

3. $f(x) = \frac{1+x^2}{1-x^2}$

4. $f(x) = \frac{1}{1+4x^2}$

5. $f(x) = \frac{1}{x^4+16}$

6. $f(x) = \frac{x}{x-3}$

7. $f(x) = \frac{2}{3x+4}$

8–9 Express the function as the sum of a power series by first using partial fractions. Find the interval of convergence.

8. $f(x) = \frac{3x-2}{2x^2-3x+1}$

9. $f(x) = \frac{x}{x^2-3x+2}$

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10–11 Find a power series representation for the function and determine the radius of convergence.

10. $f(x) = \tan^{-1}(2x)$

11. $f(x) = \ln\left(\frac{1+x}{1-x}\right)$

12–14 Evaluate the indefinite integral as a power series.

12. $\int \frac{1}{1+x^4} dx$

13. $\int \frac{x}{1+x^5} dx$

14. $\int \frac{\arctan x}{x} dx$

15–16 Use a power series to approximate the definite integral to six decimal places.

15. $\int_0^{0.2} \frac{1}{1+x^4} dx$

16. $\int_0^{1/2} \tan^{-1}(x^2) dx$

Answers

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$$1. \sum_{n=1}^{\infty} x^n, (-1, 1)$$

$$2. \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^{n+1}}, (-2, 2)$$

$$3. 1 + \sum_{n=1}^{\infty} 2x^{2n}, (-1, 1)$$

$$4. \sum_{n=0}^{\infty} (-1)^n 4^n x^{2n}, \left(-\frac{1}{2}, \frac{1}{2}\right)$$

$$5. \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{2^{4n+4}}, (-2, 2)$$

$$6. -\sum_{n=1}^{\infty} \left(\frac{x}{3}\right)^n, (-3, 3)$$

$$7. \sum_{n=0}^{\infty} \frac{(-1)^n 3^n x^n}{2^{2n+1}}, \left(-\frac{4}{3}, \frac{4}{3}\right)$$

$$8. -\sum_{n=0}^{\infty} (2^n + 1)x^n, \left(-\frac{1}{2}, \frac{1}{2}\right)$$

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$$9. \sum_{n=0}^{\infty} (1 - 2^{-n})x^n, (-1, 1)$$

$$10. \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{2n+1}, \frac{1}{2}$$

$$11. \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1}, 1$$

$$12. C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{4n+1}$$

$$13. C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{5n+2}}{5n+2}$$

$$14. C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)^2}$$

$$15. 0.199936$$

$$16. 0.041303$$

Solutions

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“ R ” stands for “radius of convergence” and “ I ” stands for “interval of convergence” in this section.

$$1. f(x) = \frac{x}{1-x} = x \left(\frac{1}{1-x} \right) = x \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+1} \\ = \sum_{n=1}^{\infty} x^n \\ \text{with } R = 1 \text{ and } I = (-1, 1).$$

$$2. f(x) = \frac{1}{4+x^2} = \frac{1}{4} \left(\frac{1}{1+x^2/4} \right) \\ = \frac{1}{4} \sum_{n=0}^{\infty} (-1)^n \left(\frac{x^2}{4} \right)^n \quad \left(\begin{array}{l} \text{using Exercise 3} \\ \text{from the text} \end{array} \right) \\ = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^{n+1}}$$

$$\text{with } \left| \frac{x^2}{4} \right| < 1 \Leftrightarrow x^2 < 4 \Leftrightarrow |x| < 2, \text{ so } R = 2 \text{ and} \\ I = (-2, 2).$$

$$3. f(x) = \frac{1+x^2}{1-x^2} = 1 + \frac{2x^2}{1-x^2} = 1 + 2x^2 \sum_{n=0}^{\infty} (x^2)^n \\ = 1 + \sum_{n=0}^{\infty} 2x^{2n+2} = 1 + \sum_{n=1}^{\infty} 2x^{2n} \\ \text{with } |x^2| < 1 \Leftrightarrow |x| < 1, \text{ so } R = 1 \text{ and } I = (-1, 1).$$

$$4. f(x) = \frac{1}{1+4x^2} = \sum_{n=0}^{\infty} (-1)^n (4x^2)^n \quad \left(\begin{array}{l} \text{using Exercise 3} \\ \text{from the text} \end{array} \right) \\ = \sum_{n=0}^{\infty} (-1)^n 4^n x^{2n} \\ \text{with } |4x^2| < 1 \text{ so } x^2 < \frac{1}{4} \Leftrightarrow |x| < \frac{1}{2}, \text{ and so } R = \frac{1}{2} \text{ and} \\ I = \left(-\frac{1}{2}, \frac{1}{2}\right).$$

$$5. f(x) = \frac{1}{x^4+16} = \frac{1}{16} \left[\frac{1}{1+(x/2)^4} \right] \\ = \frac{1}{16} \sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{2} \right)^{4n} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{2^{4n+4}} \\ \text{for } \left| \frac{x}{2} \right| < 1 \Leftrightarrow |x| < 2 \text{ so, } R = 2 \text{ and } I = (-2, 2).$$

$$6. f(x) = \frac{x}{x-3} = 1 + \frac{3}{x-3} = 1 - \frac{1}{1-x/3} \\ = 1 - \sum_{n=0}^{\infty} \left(\frac{x}{3} \right)^n = - \sum_{n=1}^{\infty} \left(\frac{x}{3} \right)^n \\ \text{For convergence, } \left| \frac{x}{3} \right| < 1 \Leftrightarrow |x| < 3, \text{ so } R = 3 \text{ and} \\ I = (-3, 3).$$

Another Method:

$$f(x) = \frac{x}{x-3} = - \frac{x}{3(1-x/3)} = - \frac{x}{3} \sum_{n=0}^{\infty} \left(\frac{x}{3} \right)^n \\ = - \sum_{n=0}^{\infty} \frac{x^{n+1}}{3^{n+1}} = - \sum_{n=1}^{\infty} \frac{x^n}{3^n}$$

$$7. f(x) = \frac{2}{3x+4} = \frac{1}{2} \left(\frac{1}{1+3x/4} \right) \\ = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \left(\frac{3x}{4} \right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n 3^n x^n}{2^{2n+1}} \\ \left| \frac{3x}{4} \right| < 1 \text{ so } R = \frac{4}{3} \text{ and } I = \left(-\frac{4}{3}, \frac{4}{3}\right).$$

$$8. \frac{3x-2}{2x^2-3x+1} = \frac{3x-2}{(2x-1)(x-1)} = \frac{A}{2x-1} + \frac{B}{x-1} \\ \Leftrightarrow A+2B=3 \text{ and } -A-B=-2 \Leftrightarrow A=B=1, \text{ so} \\ f(x) = \frac{3x-2}{2x^2-3x+1} = \frac{1}{2x-1} + \frac{1}{x-1} \\ = - \sum_{n=0}^{\infty} (2x)^{-n} - \sum_{n=0}^{\infty} x^{-n} = - \sum_{n=0}^{\infty} (2^n+1)x^{-n} \\ \text{with } R = \frac{1}{2}. \text{ At } x = \pm \frac{1}{2}, \text{ the series diverges by the Test for} \\ \text{Divergence, so } I = \left(-\frac{1}{2}, \frac{1}{2}\right).$$

$$9. \frac{x}{x^2-3x+2} = \frac{x}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1} \Leftrightarrow \\ A+B=1 \text{ and } -A-2B=0 \Leftrightarrow A=2, B=-1, \text{ so} \\ f(x) = \frac{x}{(x-2)(x-1)} = \frac{2}{x-2} - \frac{1}{x-1} \\ = - \frac{1}{1-x/2} + \frac{1}{1-x} = - \sum_{n=0}^{\infty} \left(\frac{x}{2} \right)^n + \sum_{n=0}^{\infty} x^n \\ = \sum_{n=0}^{\infty} (1-2^{-n})x^n \\ \text{with } R = 1. \text{ At } x = \pm 1, \text{ the series diverges by the Test for} \\ \text{Divergence, so } I = (-1, 1).$$

$$10. f(x) = \tan^{-1} 2x = 2 \int \frac{dx}{1+4x^2} \\ = 2 \int \sum_{n=0}^{\infty} (-1)^n (4x^2)^n dx \\ = 2 \int \sum_{n=0}^{\infty} (-1)^n 4^n x^{2n} dx \\ = C + 2 \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n+1}}{2n+1} \\ = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{2n+1} \\ \text{for } |4x^2| < 1 \text{ so } |x| < \frac{1}{2} \text{ and } R = \frac{1}{2}.$$

$$\begin{aligned}
 11. \quad f(x) &= \ln(1+x) - \ln(1-x) = \int \frac{dx}{1+x} + \int \frac{dx}{1-x} \\
 &= \int \left[\sum_{n=0}^{\infty} (-1)^n x^n + \sum_{n=0}^{\infty} x^n \right] dx \\
 &= \int \sum_{n=0}^{\infty} 2x^{2n} dx = \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1} + C
 \end{aligned}$$

But $f(0) = \ln 1 - \ln 1 = 0$, so $C = 0$ and we have

$$f(x) = \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1} \text{ with } R = 1.$$

$$12. \quad \int \frac{dx}{1+x^4} = \int \sum_{n=0}^{\infty} (-1)^n x^{4n} dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{4n+1}$$

with $R = 1$.

$$13. \quad \frac{1}{1+x^5} = \sum_{n=0}^{\infty} (-1)^n x^{5n} \Rightarrow$$

$$\frac{x}{1+x^5} = \sum_{n=0}^{\infty} (-1)^n x^{5n+1} \Rightarrow$$

$$\int \frac{x}{1+x^5} dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{5n+2}}{5n+2} \text{ with } R = 1.$$

$$14. \quad \text{By Example 7, } \arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \text{ so}$$

$$\begin{aligned}
 \int \frac{\arctan x}{x} dx &= \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n+1} dx \\
 &= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)^2}
 \end{aligned}$$

with $R = 1$.

15. We use the representation

$$\int \frac{dx}{1+x^4} = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{4n+1} \text{ from}$$

Problem 12 with $C = 0$. So

$$\begin{aligned}
 \int_0^{0.2} \frac{dx}{1+x^4} &= \left[x - \frac{x^5}{5} + \frac{x^9}{9} - \frac{x^{13}}{13} + \dots \right]_0^{0.2} \\
 &= 0.2 - \frac{0.2^5}{5} + \frac{0.2^9}{9} - \frac{0.2^{13}}{13} + \dots
 \end{aligned}$$

Since the series is alternating, the error in the n th-order approximation is less than the first neglected term, by The Alternating Series Estimation Theorem. If we use only the first two terms of the series, then the error is at most

$$0.2^9/9 \approx 5.7 \times 10^{-8}. \text{ So, to six decimal places,}$$

$$\int_0^{0.2} \frac{dx}{1+x^4} \approx 0.2 - \frac{0.2^5}{5} = 0.199936.$$

16. We use the representation

$$\int \tan^{-1}(x^2) dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(2n+1)(4n+3)}$$

from Exercise 26 in the text with $C = 0$:

$$\begin{aligned}
 \int_0^{1/2} \tan^{-1}(x^2) dx &= \left[\frac{x^3}{3} - \frac{x^7}{21} + \frac{x^{11}}{55} - \frac{x^{15}}{105} + \frac{x^{19}}{171} - \dots \right]_0^{1/2} \\
 &= \frac{0.5^3}{3} - \frac{0.5^7}{21} + \frac{0.5^{11}}{55} - \frac{0.5^{15}}{105} + \frac{0.5^{19}}{171} - \dots
 \end{aligned}$$

The series is alternating, so if we use only the first four terms of the series, then the error is at most

$$0.5^{19}/171 \approx 1.1 \times 10^{-8}. \text{ So, to six decimal places,}$$

$$\begin{aligned}
 \int_0^{1/2} \tan^{-1}(x^2) dx &\approx \frac{1}{3} (0.5)^3 - \frac{1}{21} (0.5)^7 + \frac{1}{55} (0.5)^{11} - \frac{1}{105} (0.5)^{15} \\
 &\approx 0.041303
 \end{aligned}$$