

 Answers

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1. 1, $[-1, 1)$
2. 1, $(-1, 1]$
3. 2, $(-2, 2]$
4. $\frac{1}{5}$, $(-\frac{1}{5}, \frac{1}{5})$
5. 1, $(-1, 1)$
6. 1, $[-1, 1]$
7. $\frac{1}{3}$, $[-\frac{1}{3}, \frac{1}{3}]$
8. 10, $(-10, 10)$
9. 1, $[-1, 1)$
10. ∞ , $(-\infty, \infty)$

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11. $\frac{1}{2}$, $(\frac{5}{2}, \frac{7}{2})$
12. 1, $[-2, 0]$
13. $\frac{1}{3}$, $(-1, -\frac{1}{3})$
14. 2, $(-\frac{3}{2}, \frac{5}{2})$
15. 1, $(0, 2]$
16. 5, $[-1, 9)$
17. $\frac{1}{3}$, $(\frac{2}{3}, \frac{4}{3}]$
18. $\frac{1}{2}$, $[0, 1]$
19. ∞ , $(-\infty, \infty)$

Solutions

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“ R ” stands for “radius of convergence” and “ I ” stands for “interval of convergence” in this section.

1. If $a_n = \frac{x^n}{n+2}$, then

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{n+3} \cdot \frac{n+2}{x^n} \right| \\ &= |x| \lim_{n \rightarrow \infty} \frac{n+2}{n+3} = |x| < 1\end{aligned}$$

for convergence (by the Ratio Test). So $R = 1$. When $x = 1$,

the series is $\sum_{n=0}^{\infty} \frac{1}{n+2}$ which diverges (Integral Test or

Comparison Test), and when $x = -1$, it is $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+2}$ which

converges (Alternating Series Test), so $I = [-1, 1)$.

2. If $a_n = \frac{(-1)^n x^n}{\sqrt[3]{n}}$, then

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= |x| \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^{1/3} = |x| < 1 \text{ for} \\ \text{convergence (by the Ratio Test), and } R &= 1. \text{ When } x = 1, \\ \sum_{n=1}^{\infty} a_n &= \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}} \text{ which is a convergent alternating}\end{aligned}$$

series, but when $x = -1$, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ which is a

divergent p -series ($p = \frac{1}{3} < 1$), so $I = (-1, 1]$.

3. If $a_n = \frac{(-1)^n x^n}{n2^n}$, then

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1} / [(n+1)2^{n+1}]}{x^n / (n2^n)} \right| \\ &= \left| \frac{x}{2} \right| \lim_{n \rightarrow \infty} \frac{n}{n+1} = \left| \frac{x}{2} \right| < 1\end{aligned}$$

for convergence, so $|x| < 2$ and $R = 2$. When $x = 2$,

$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which converges by

the Alternating Series Test. When $x = -2$,

$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ which diverges (harmonic series),

so $I = (-2, 2]$.

4. If $a_n = n5^n x^n$, then

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 5|x| \lim_{n \rightarrow \infty} \frac{n+1}{n} = 5|x| < 1 \text{ for}$$

convergence (by the Ratio Test), so $R = \frac{1}{5}$. If $x = \pm \frac{1}{5}$,

$|a_n| = n \rightarrow \infty$ as $n \rightarrow \infty$, so $\sum_{n=1}^{\infty} a_n$ diverges by the Test

for Divergence and $I = (-\frac{1}{5}, \frac{1}{5})$.

5. If $a_n = nx^n$, then

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)x^{n+1}}{nx^n} \right| \\ &= |x| \lim_{n \rightarrow \infty} \frac{n+1}{n} = |x| < 1\end{aligned}$$

for convergence (by the Ratio Test). So $R = 1$. When

$x = 1$ or -1 , $\lim_{n \rightarrow \infty} nx^n$ does not exist, so $\sum_{n=0}^{\infty} nx^n$ diverges

for $x = \pm 1$. So $I = (-1, 1)$.

6. If $a_n = \frac{x^n}{n^2}$ then

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = |x| \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^2 = |x| < 1 \text{ for}$$

convergence (by the Ratio Test), so $R = 1$. If $x = \pm 1$,

$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges ($p = 2 > 1$), so

$I = [-1, 1]$.

7. If $a_n = \frac{3^n x^n}{(n+1)^2}$, then

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{3^{n+1} x^{n+1}}{(n+2)^2} \cdot \frac{(n+1)^2}{3^n x^n} \right| \\ &= 3|x| \lim_{n \rightarrow \infty} \left(\frac{n+1}{n+2} \right)^2 = 3|x| < 1\end{aligned}$$

for convergence, so $|x| < \frac{1}{3}$ and $R = \frac{1}{3}$. When $x = \frac{1}{3}$,

$\sum_{n=0}^{\infty} \frac{3^n x^n}{(n+1)^2} = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$, which is a

convergent p -series ($p = 2 > 1$). When $x = -\frac{1}{3}$,

$\sum_{n=0}^{\infty} \frac{3^n x^n}{(n+1)^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2}$, which converges by the

Alternating Series Test, so $I = [-\frac{1}{3}, \frac{1}{3}]$.

8. If $a_n = \frac{n^2 x^n}{10^n}$, then

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{|x|}{10} \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^2 = \frac{|x|}{10} < 1 \text{ for}$$

convergence (by the Ratio Test), so $R = 10$. If $x = \pm 10$,

$|a_n| = n^2 \rightarrow \infty$ as $n \rightarrow \infty$, so $\sum_{n=0}^{\infty} a_n$ diverges (Test for

Divergence) and $I = (-10, 10)$.

9. If $a_n = \frac{x^n}{\ln n}$, then

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{\ln(n+1)} \cdot \frac{\ln n}{x^n} \right| \\ &= |x| \lim_{n \rightarrow \infty} \frac{\ln n}{\ln(n+1)} \stackrel{H}{=} |x| \end{aligned}$$

so $R = 1$. When $x = 1$, $\sum_{n=2}^{\infty} \frac{x^n}{\ln n} = \sum_{n=2}^{\infty} \frac{1}{\ln n}$, which

diverges because $\frac{1}{\ln n} > \frac{1}{n}$ and $\sum_{n=2}^{\infty} \frac{1}{n}$ is the divergent

harmonic series. When $x = -1$, $\sum_{n=2}^{\infty} \frac{x^n}{\ln n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$,

which converges by the Alternating Series Test.

So $I = [-1, 1)$.

10. If $a_n = \frac{(-1)^n x^{2n-1}}{(2n-1)!}$, then

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{x^2}{(2n+1)2n} = 0 < 1 \text{ for all } x. \text{ By}$$

the Ratio Test the series converges for all x , so $R = \infty$ and $I = (-\infty, \infty)$.

11. If $a_n = \frac{2^n (x-3)^n}{n+3}$, then

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{2^{n+1} (x-3)^{n+1}}{n+4} \cdot \frac{n+3}{2^n (x-3)^n} \right| \\ &= 2|x-3| \lim_{n \rightarrow \infty} \frac{n+3}{n+4} = 2|x-3| < 1 \end{aligned}$$

for convergence, or $|x-3| < \frac{1}{2} \Leftrightarrow \frac{5}{2} < x < \frac{7}{2}$, and

$R = \frac{1}{2}$. When $x = \frac{5}{2}$, $\sum_{n=0}^{\infty} \frac{2^n (x-3)^n}{n+3} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+3}$,

which converges by the Alternating Series Test. When

$x = \frac{7}{2}$, $\sum_{n=0}^{\infty} \frac{2^n (x-3)^n}{n+3} = \sum_{n=0}^{\infty} \frac{1}{n+3}$, similar to the

harmonic series, which diverges. So $I = [\frac{5}{2}, \frac{7}{2})$.

12. If $a_n = \frac{(x+1)^n}{n(n+1)}$, then

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = |x+1| \lim_{n \rightarrow \infty} \frac{n}{n+2} = |x+1| < 1 \text{ for}$$

convergence, or $-2 < x < 0$ and $R = 1$. If $x = -2$ or 0,

then $|a_n| = \frac{1}{n^2 + n} < \frac{1}{n^2}$, so $\sum_{n=1}^{\infty} |a_n|$ converges since

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ does ($p = 2 > 1$), and $I = [-2, 0]$.

13. If $a_n = \sqrt{n} (3x+2)^n$, then

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{\sqrt{n+1} (3x+2)^{n+1}}{\sqrt{n} (3x+2)^n} \right| \\ &= \left| \sqrt{1 + \frac{1}{n}} \cdot (3x+2) \right| \rightarrow |3x+2| \text{ as } n \rightarrow \infty \end{aligned}$$

so for convergence, $|3x+2| < 1 \Rightarrow |x + \frac{2}{3}| < \frac{1}{3}$ so

$R = \frac{1}{3}$ and $-1 < x < -\frac{1}{3}$. If $x = -1$, the series becomes

$\sum_{n=0}^{\infty} (-1)^n \sqrt{n}$ which is divergent by the Test for Divergence.

If $x = -\frac{1}{3}$, the series is $\sum_{n=0}^{\infty} \sqrt{n}$ which is also divergent by the Test for Divergence. So $I = (-1, -\frac{1}{3})$.

14. If $a_n = \frac{n}{4^n} (2x-1)^n$, then

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{(n+1)(2x-1)^{n+1}}{4^{n+1}} \cdot \frac{4^n}{n(2x-1)^n} \right| \\ &= \left| \frac{2x-1}{4} \left(1 + \frac{1}{n}\right) \right| \rightarrow \frac{1}{2} |x - \frac{1}{2}| \text{ as } n \rightarrow \infty. \end{aligned}$$

For convergence, $\frac{1}{2} |x - \frac{1}{2}| < 1 \Rightarrow |x - \frac{1}{2}| < 2 \Rightarrow$

$R = 2$ and $-2 < x - \frac{1}{2} < 2 \Rightarrow -\frac{3}{2} < x < \frac{5}{2}$. If

$x = -\frac{3}{2}$, the series becomes $\sum_{n=0}^{\infty} \frac{n}{4^n} (-4)^n = \sum_{n=0}^{\infty} (-1)^n n$

which is divergent by the Test for Divergence. If $x = \frac{5}{2}$, the

series is $\sum_{n=0}^{\infty} \frac{n}{4^n} 4^n = \sum_{n=0}^{\infty} n$, also divergent by the Test for

Divergence. So $I = (-\frac{3}{2}, \frac{5}{2})$.

15. If $a_n = \frac{(-1)^n (x-1)^n}{\sqrt{n}}$, then

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(x-1)^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(x-1)^n} \right| \\ &= |x-1| \lim_{n \rightarrow \infty} \sqrt{\frac{n}{n+1}} = |x-1| < 1 \end{aligned}$$

for convergence, or $0 < x < 2$, and $R = 1$. When $x = 0$,

$\sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which is a divergent

p -series ($p = \frac{1}{2} < 1$). When $x = 2$, the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

which converges by the Alternating Series Test. So

$I = (0, 2]$.

16. If $a_n = \frac{(x-4)^n}{n5^n}$ then

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{|x-4|}{5} \lim_{n \rightarrow \infty} \frac{n}{n+1} = \frac{|x-4|}{5} < 1 \text{ for}$$

convergence, or $-1 < x < 9$ and $R = 5$. When $x = 9$,

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n}$ which diverges (harmonic series), and when

$x = -1$, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which converges by the

Alternating Series Test, so $I = [-1, 9)$.

17. If $a_n = \frac{(-3)^n (x-1)^n}{\sqrt{n+1}}$ then

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 3|x-1| \lim_{n \rightarrow \infty} \left(\frac{n+1}{n+2} \right)^{1/2} \\ = 3|x-1| < 1$$

for convergence, or $\frac{2}{3} < x < \frac{4}{3}$ and $R = \frac{1}{3}$. When $x = \frac{4}{3}$,

$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ which is a convergent alternating series, and when $x = \frac{2}{3}$, $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$ which is a divergent p -series ($p = \frac{1}{2} < 1$), so $I = \left(\frac{2}{3}, \frac{4}{3}\right]$.

18. If $a_n = \frac{(2x-1)^n}{n^3}$, then

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = |2x-1| \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^3 = |2x-1| < 1$$

for convergence, so $|x - \frac{1}{2}| < \frac{1}{2} \Leftrightarrow 0 < x < 1$, and

$R = \frac{1}{2}$. The series $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{n^3}$ converges both for $x = 0$ and $x = 1$ (in the first case because of the Alternating Series Test and in the second case because we get a p -series with $p = 3 > 1$). So $I = [0, 1]$.

19. If $a_n = \frac{nx^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$, then

$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = |x| \lim_{n \rightarrow \infty} \frac{n+1}{n(2n+1)} = 0$ for all x . So the series converges for all $x \Rightarrow R = \infty$ and $I = (-\infty, \infty)$.