8.4 Other Convergence Tests

A Click here for answers.

S Click here for solutions.

1–14 Test the series for convergence or divergence.

1. $\frac{3}{5} - \frac{3}{6} + \frac{3}{7} - \frac{3}{8} + \frac{3}{9} - \cdots$

2. $-5 - \frac{5}{2} + \frac{5}{5} - \frac{5}{8} + \frac{5}{11} - \frac{5}{14} + \cdots$

3. $-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \frac{6}{7} - \cdots$

4. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$

5. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+3}}$

6. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{5n+1}$

7. $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n \ln n}$

8. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$

9. $\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^2+1}$

10. $\sum_{n=1}^{\infty} (-1)^n \frac{2n}{4n+1}$

11. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n^2}{4n^2+1}$

12. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt{n}}{n+4}$

13. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{2^n}$

14. $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt[3]{\ln n}}$

15–18 Approximate the sum of the series to the indicated accuracy.

15. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!}$ (four decimal places)

16. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}$ (four decimal places)

17. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!}$ (four decimal places)

18. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^6}$ (five decimal places)

19–38 Determine whether the series is absolutely convergent.

19. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\sqrt{n}}$

20. $\sum_{n=1}^{\infty} \frac{(-3)^n}{n^3}$

21. $\sum_{n=0}^{\infty} \frac{(-3)^n}{n!}$

22. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}$

23. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2+1}$

24. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!}$

25. $\sum_{n=1}^{\infty} (-1)^n \frac{2n}{3n-4}$

26. $\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n^2+1}$

27. $\sum_{n=1}^{\infty} \frac{(-2)^n}{n 3^{n+1}}$

28. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 5^{n-1}}{(n+1)^2 4^{n+2}}$

29. $\sum_{n=1}^{\infty} \frac{(n+1)5^n}{n 3^{2n}}$

30. $\sum_{n=1}^{\infty} \frac{\sin 2n}{n^2}$

31. $\sum_{n=1}^{\infty} \frac{(-1)^n \arctan n}{n^3}$

32. $\sum_{n=1}^{\infty} \frac{\cos(n\pi/6)}{n\sqrt{n}}$

33. $\sum_{n=1}^{\infty} \frac{n!}{(-10)^n}$

34. $\sum_{n=1}^{\infty} \frac{8-n^3}{n!}$

35. $\sum_{n=1}^{\infty} \frac{(-n)^n}{5^{2n+3}}$

36. $\sum_{n=1}^{\infty} \frac{(-2)^n n^2}{(n+2)!}$

37. $\sum_{n=1}^{\infty} \frac{(n+2)!}{n!\,10^n}$

38. $$\frac{1}{3} + \frac{1 \cdot 4}{3 \cdot 5} + \frac{1 \cdot 4 \cdot 7}{3 \cdot 5 \cdot 7} + \frac{1 \cdot 4 \cdot 7 \cdot 10}{3 \cdot 5 \cdot 7 \cdot 9} + \cdots + \frac{1 \cdot 4 \cdot 7 \cdot \cdots \cdot (3n-2)}{3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n+1)} + \cdots$$

Answers

E Click here for exercises.

S Click here for solutions.

1. Convergent
2. Convergent
3. Divergent
4. Convergent
5. Convergent
6. Divergent
7. Convergent
8. Convergent
9. Divergent
10. Divergent
11. Divergent
12. Convergent
13. Convergent
14. Convergent
15. 0.8415
16. 0.5403
17. 0.6065
18. 0.98555
19. Absolutely convergent
20. Divergent
21. Absolutely convergent
22. Absolutely convergent
23. Absolutely convergent
24. Absolutely convergent
25. Divergent
26. Divergent
27. Absolutely convergent
28. Divergent
29. Absolutely convergent
30. Absolutely convergent
31. Absolutely convergent
32. Absolutely convergent
33. Divergent
34. Absolutely convergent
35. Divergent
36. Absolutely convergent
37. Absolutely convergent
38. Divergent

Solutions

E Click here for exercises.

A Click here for answers.

1. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{3}{n+4}$. $b_n=\frac{3}{n+4}>0$ and $b_{n+1}<b_n$ for all n; $\lim_{n\to\infty}b_n=0$ so the series converges by the Alternating Series Test.

2. $-5+\sum_{n=0}^{\infty}(-1)^{n-1}\frac{5}{3n+2}$. $b_n=\frac{5}{3n+2}$ is decreasing and positive for all n, and $\lim_{n\to\infty}\frac{5}{3n+2}=0$ so the series converges by the Alternating Series Test.

3. $\sum_{n=1}^{\infty}(-1)^n\frac{n}{n+1}$. $\lim_{n\to\infty}\frac{n}{n+1}=1$ so $\lim_{n\to\infty}(-1)^n\frac{n}{n+1}$ does not exist and the series diverges by the Test for Divergence.

4. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2}$. $b_n=\frac{1}{n^2}>0$ and $b_{n+1}<b_n$ for all n, and $\lim_{n\to\infty}\frac{1}{n^2}=0$, so the series converges by the Alternating Series Test.

5. $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n+3}}$. $b_n=\frac{1}{\sqrt{n+3}}$ is positive and decreasing, and $\lim_{n\to\infty}\frac{1}{\sqrt{n+3}}=0$, so the series converges by the Alternating Series Test.

6. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{5n+1}$. $\lim_{n\to\infty}\frac{n}{5n+1}=\frac{1}{5}$ so $\lim_{n\to\infty}(-1)^{n+1}\frac{n}{5n+1}$ does not exist and the series diverges by the Test for Divergence.

7. $\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{n\ln n}$. $b_n=\frac{1}{n\ln n}$ is positive and decreasing for $n\geq 2$, and $\lim_{n\to\infty}\frac{1}{n\ln n}=0$ so the series converges by the Alternating Series Test.

8. $\sum_{n=1}^{\infty}(-1)^n\frac{n}{n^2+1}$. $b_n=\frac{n}{n^2+1}>0$ for all n.

$b_{n+1}<b_n \quad\Leftrightarrow\quad \frac{n+1}{(n+1)^2+1}<\frac{n}{n^2+1} \quad\Leftrightarrow$

$(n+1)\left(n^2+1\right)<\left[(n+1)^2+1\right]n \quad\Leftrightarrow$

$n^3+n^2+n+1<n^3+2n^2+2n \quad\Leftrightarrow$

$0<n^2+n-1$, which is true for all $n\geq 1$. Also $\lim_{n\to\infty}\frac{n}{n^2+1}=\lim_{n\to\infty}\frac{1/n}{1+1/n^2}=0$. Therefore the series converges by the Alternating Series Test.

9. $\sum_{n=1}^{\infty}(-1)^n\frac{n^2}{n^2+1}$. $\lim_{n\to\infty}\frac{n^2}{n^2+1}=1$, so $\lim_{n\to\infty}(-1)^n\frac{n^2}{n^2+1}$ does not exist. Thus the series diverges by the Test for Divergence.

10. $a_n=(-1)^n\frac{2n}{4n+1}$, so $|a_n|=\frac{2n}{4n+1}\to\frac{1}{2}$ as $n\to\infty$. Therefore, $\lim_{n\to\infty}a_n\neq 0$ (in fact the limit does not exist) and the series $\sum_{n=1}^{\infty}(-1)^n\frac{2n}{4n+1}$ diverges by the Test for Divergence.

11. $a_n=(-1)^{n-1}\frac{2n^2}{4n^2+1}$, so $|a_n|=\frac{2n^2}{4n^2+1}\to\frac{1}{2}$ as $n\to\infty$. Therefore, $\lim_{n\to\infty}a_n\neq 0$ (in fact the limit does not exist) and the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2n^2}{4n^2+1}$ diverges by the Test for Divergence.

12. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sqrt{n}}{n+4}$. $b_n=\frac{\sqrt{n}}{n+4}>0$ for all n. Let $f(x)=\frac{\sqrt{x}}{x+4}$. Then $f'(x)=\frac{4-x}{2\sqrt{x}\,(x+4)^2}<0$ if $x>4$, so $\{b_n\}$ is decreasing after $n=4$.

$\lim_{n\to\infty}\frac{\sqrt{n}}{n+4}=\lim_{n\to\infty}\frac{1}{\sqrt{n}+4/\sqrt{n}}=0$. So the series converges by the Alternating Series Test.

13. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{2^n}$. $b_n=\frac{n}{2^n}>0$ and $b_n\geq b_{n+1} \quad\Leftrightarrow$

$\frac{n}{2^n}\geq\frac{n+1}{2^{n+1}} \quad\Leftrightarrow\quad 2n\geq n+1 \quad\Leftrightarrow\quad n\geq 1$ which is certainly true. $\lim_{n\to\infty}(n/2^n)=0$ by l'Hospital's Rule, so the series converges by the Alternating Series Test.

14. $\frac{1}{\sqrt[3]{\ln n}}$ decreases and $\lim_{n\to\infty}\frac{1}{\sqrt[3]{\ln n}}=0$, so by the Alternating Series Test the series converges.

15. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)!}$. $b_5=\frac{1}{(2\cdot 5-1)!}=\frac{1}{362{,}880}<0.00001$, so $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)!}\approx\sum_{n=1}^{4}\frac{(-1)^{n-1}}{(2n-1)!}\approx 0.8415$.

16. $b_4=\frac{1}{(2\cdot 4)!}=\frac{1}{40{,}320}\approx 0.000025$ and $s_3=1-\frac{1}{2}+\frac{1}{24}-\frac{1}{720}\approx 0.54028$, so, correct to four decimal places, $\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\approx 0.5403$.

17. $b_6 = \dfrac{1}{2^6 6!} = \dfrac{1}{46{,}080} \approx 0.000022 < 0.0001$, so
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} \approx \sum_{n=0}^{5} \frac{(-1)^n}{2^n n!} \approx 0.6065.$$

18. $b_8 = 1/8^6 \approx 0.0000038 < 0.00001$ and
$$s_7 = 1 - \tfrac{1}{64} + \tfrac{1}{729} - \tfrac{1}{4096} + \tfrac{1}{15{,}625} - \tfrac{1}{46{,}656} + \tfrac{1}{117{,}649} \approx 0.9855537$$
so correct to five decimal places, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^6} \approx 0.98555$.

19. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series ($p = \frac{3}{2} > 1$), so the given series is absolutely convergent.

20. Using the Ratio Test,
$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-3)^{n+1} / (n+1)^3}{(-3)^n / n^3} \right| = 3 \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^3 = 3 > 1$$
so the series diverges.

21. Using the Ratio Test, $\displaystyle\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-3)^{n+1} / (n+1)!}{(-3)^n / n!} \right| = 3 \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$, so the series is absolutely convergent.

22. $\displaystyle\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$, so the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}$ is absolutely convergent by the Ratio Test.

23. $\dfrac{1}{n^2+1} < \dfrac{1}{n^2}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges ($p = 2 > 1$), so $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges absolutely by the Comparison Test.

24.
$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{1/[(2n+1)!]}{1/[(2n-1)!]} = \lim_{n\to\infty} \frac{1}{(2n+1)\,2n} = 0$$
so by the Ratio Test the series is absolutely convergent.

25. $\displaystyle\lim_{n\to\infty} \frac{2n}{3n-4} = \frac{2}{3}$, so $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{2n}{3n-4}$ diverges by the Test for Divergence.

26. $\displaystyle\lim_{n\to\infty} (-1)^n \frac{2^n}{n^2+1}$ does not exist, so $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n^2+1}$ diverges by the Test for Divergence.

27.
$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{2^{n+1} / [(n+1)\, 3^{n+2}]}{2^n / (n3^{n+1})} \right| = \frac{2}{3} \lim_{n\to\infty} \frac{n}{n+1} = \frac{2}{3} < 1$$
so the series converges absolutely by the Ratio Test.

28.
$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{5^n / [(n+2)^2\, 4^{n+3}]}{5^{n-1} / [(n+1)^2\, 4^{n+2}]} \right| = \frac{5}{4} \lim_{n\to\infty} \left(\frac{n+1}{n+2} \right)^2 = \frac{5}{4} > 1$$
so the series diverges by the Ratio Test.

29.
$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+2)\, 5^{n+1} \Big/ \left[(n+1)\, 3^{2(n+1)}\right]}{(n+1)\, 5^n / (n3^{2n})} = \lim_{n\to\infty} \frac{5n\,(n+2)}{9\,(n+1)^2} = \frac{5}{9} < 1$$
so the series converges absolutely by the Ratio Test.

30. $\left| \dfrac{\sin 2n}{n^2} \right| \le \dfrac{1}{n^2}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series, $p = 2 > 1$), so $\displaystyle\sum_{n=1}^{\infty} \frac{\sin 2n}{n^2}$ converges absolutely by the Comparison Test.

31. $\dfrac{\arctan n}{n^3} < \dfrac{\pi/2}{n^3}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{\pi/2}{n^3}$ converges ($p = 3 > 1$), so $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\arctan n}{n^3}$ converges absolutely by the Comparison Test.

32. $\left| \cos \dfrac{n\pi}{6} \right| \le 1$, so since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ converges ($p = \frac{3}{2} > 1$), the given series converges absolutely by the Comparison Test.

33.
$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+1)!/10^{n+1}}{n!/10^n} = \lim_{n\to\infty} \frac{n+1}{10} = \infty$$
so the series diverges by the Ratio Test.

34.
$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\left[8 - (n+1)^3\right] / [(n+1)!]}{(8 - n^3)/(n!)} \right| = \lim_{n\to\infty} \frac{1}{n+1} \left| \frac{8 - (n+1)^3}{8 - n^3} \right| = 0 < 1$$
so the series converges absolutely by the Ratio Test.

35.
$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+1)^{n+1} / 5^{2n+5}}{n^n / 5^{2n+3}} = \lim_{n\to\infty} \frac{1}{25} \left(\frac{n+1}{n} \right)^n (n+1) = \infty$$
so the series diverges by the Ratio Test.

36. $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-2)^{n+1}(n+1)^2/[(n+3)!]}{(-2)^n n^2/[(n+2)!]}\right| = \lim_{n\to\infty}\frac{2(n+1)^2}{n^2(n+3)} = 0 < 1$$

so the series converges absolutely by the Ratio Test.

37. $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+3)!\,/\left[(n+1)!10^{n+1}\right]}{(n+2)!\,/(n!10^n)} = \frac{1}{10}\lim_{n\to\infty}\frac{n+3}{n+1} = \frac{1}{10} < 1$$

so the series converges absolutely by the Ratio Test.

38. $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{1\cdot 4\cdot 7\cdot\cdots\cdot(3n-2)(3n+1)}{3\cdot 5\cdot 7\cdot\cdots\cdot(2n+1)(2n+3)}}{\dfrac{1\cdot 4\cdot 7\cdot\cdots\cdot(3n-2)}{3\cdot 5\cdot 7\cdot\cdots\cdot(2n+1)}}\right| = \lim_{n\to\infty}\frac{3n+1}{2n+3} = \frac{3}{2} > 1$$

so the series diverges by the Ratio Test.