

## 8.3 The Integral and Comparison Tests; Estimating Sums

**A** [Click here for answers.](#)

1. Use the Integral Test to determine whether the series

$$\frac{1}{3} + \frac{1}{7} + \frac{1}{11} + \frac{1}{15} + \cdots$$

is convergent or divergent.

- 2–33** ||| Determine whether the series is convergent or divergent.

- |  |   |
|--|---|
| 2. $\sum_{n=5}^{\infty} \frac{1}{n^{1.0001}}$  | 3. $\sum_{n=1}^{\infty} n^{-0.99}$  |
| 4. $\sum_{n=1}^{\infty} \frac{2}{\sqrt[3]{n}}$ | 5. $\sum_{n=1}^{\infty} \left( \frac{2}{n\sqrt{n}} + \frac{3}{n^3} \right)$ |
| 6. $\sum_{n=5}^{\infty} \frac{1}{(n-4)^2}$     | 7. $\sum_{n=1}^{\infty} \frac{1}{2n+3}$                                     |
| 8. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$  | 9. $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$                                    |
| 10. $\sum_{n=1}^{\infty} ne^{-n^2}$            | 11. $\sum_{n=1}^{\infty} \frac{n}{2^n}$                                     |
| 12. $\sum_{n=1}^{\infty} \frac{1}{4n^2+1}$     | 13. $\sum_{n=1}^{\infty} \frac{\arctan n}{1+n^2}$                           |
| 14. $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$    | 15. $\sum_{n=1}^{\infty} \frac{1}{n^2+2n+2}$                                |

**S** [Click here for solutions.](#)

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|---|--|
| 16. $\sum_{n=1}^{\infty} \frac{1}{n^3+n^2}$               | 17. $\sum_{n=1}^{\infty} \frac{3}{4^n+5}$                      |
| 18. $\sum_{n=1}^{\infty} \frac{3}{n2^n}$                  | 19. $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n-1}}$                 |
| 20. $\sum_{n=0}^{\infty} \frac{1+5^n}{4^n}$               | 21. $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n\sqrt{n}}$           |
| 22. $\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$                | 23. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)(n+2)}}$         |
| 24. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n(n+1)(n+2)}}$ | 25. $\sum_{n=1}^{\infty} \frac{n}{(n+1)2^n}$                   |
| 26. $\sum_{n=1}^{\infty} \frac{3+\cos n}{3^n}$            | 27. $\sum_{n=1}^{\infty} \frac{5n}{2n^2-5}$                    |
| 28. $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^5+4}}$          | 29. $\sum_{n=1}^{\infty} \frac{\arctan n}{n^4}$                |
| 30. $\sum_{n=3}^{\infty} \frac{1}{n^2-4}$                 | 31. $\sum_{n=1}^{\infty} \frac{n^2+1}{n^4+1}$                  |
| 32. $\sum_{n=1}^{\infty} \frac{n+1}{n2^n}$                | 33. $\sum_{n=1}^{\infty} \frac{n^2-3n}{\sqrt[3]{n^{10}-4n^2}}$ |

 **Answers**

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**E** [Click here for exercises.](#)

1. Divergent
2. Convergent
3. Divergent
4. Divergent
5. Convergent
6. Convergent
7. Divergent
8. Divergent
9. Convergent
10. Convergent
11. Convergent
12. Convergent
13. Convergent
14. Convergent
15. Convergent
16. Converges
17. Converges

**S** [Click here for solutions.](#)

18. Converges
19. Diverges
20. Diverges
21. Converges
22. Converges
23. Converges
24. Diverges
25. Converges
26. Converges
27. Diverges
28. Converges
29. Converges
30. Converges
31. Converges
32. Converges
33. Converges

## Solutions

**E** [Click here for exercises.](#)

$$1. \frac{1}{3} + \frac{1}{7} + \frac{1}{11} + \frac{1}{15} + \cdots = \sum_{n=1}^{\infty} \frac{1}{4n-1}.$$

The function  $f(x) = \frac{1}{4x-1}$  is positive, continuous, and decreasing on  $[1, \infty)$ , so the Integral Test applies.

$$\begin{aligned} \int_1^{\infty} \frac{dx}{4x-1} &= \lim_{b \rightarrow \infty} \int_1^b \frac{dx}{4x-1} = \lim_{b \rightarrow \infty} \left[ \frac{1}{4} \ln(4x-1) \right]_1^b \\ &= \lim_{b \rightarrow \infty} \left[ \frac{1}{4} \ln(4b-1) - \frac{1}{4} \ln 3 \right] = \infty \end{aligned}$$

so the improper integral diverges, and so does the series.

2.  $\sum_{n=5}^{\infty} (1/n^{1.0001})$  is a  $p$ -series,  $p = 1.0001 > 1$ , so it converges.

3.  $\sum_{n=1}^{\infty} n^{-0.99} = \sum_{n=1}^{\infty} (1/n^{0.99})$  which diverges since  $p = 0.99 < 1$ .

4.  $\sum_{n=1}^{\infty} \frac{2}{\sqrt[3]{n}} = 2 \sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ , which is a  $p$ -series,  $p = \frac{1}{3} < 1$ , so it diverges.

5.  $\sum_{n=1}^{\infty} \left( \frac{2}{n\sqrt{n}} + \frac{3}{n^3} \right) = 2 \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} + 3 \sum_{n=1}^{\infty} \frac{1}{n^3}$ , both of which are convergent  $p$ -series because  $\frac{3}{2} > 1$  and  $3 > 1$ , so  $\sum_{n=1}^{\infty} \left( \frac{2}{n\sqrt{n}} + \frac{3}{n^3} \right)$  converges by Theorem 8 in Section 11.2.

6.  $\sum_{n=5}^{\infty} \frac{1}{(n-4)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$  is a  $p$ -series,  $p = 2 > 1$ , so it converges.

7.  $f(x) = \frac{1}{2x+3}$  is positive, continuous, and decreasing on  $[1, \infty)$ , so applying the Integral Test,

$$\int_1^{\infty} \frac{dx}{2x+3} = \lim_{t \rightarrow \infty} \left[ \frac{1}{2} \ln(2x+3) \right]_1^t = \infty \Rightarrow$$

$$\sum_{n=1}^{\infty} \frac{1}{2n+3} \text{ is divergent.}$$

8. Since  $\frac{1}{\sqrt{x+1}}$  is continuous, positive, and decreasing on  $[0, \infty)$  we can apply the Integral Test.

$$\begin{aligned} \int_1^{\infty} \frac{1}{\sqrt{x+1}} dx &= \lim_{t \rightarrow \infty} [2\sqrt{x} - 2 \ln(\sqrt{x+1})]_1^t \\ &\text{[using the substitution } u = \sqrt{x+1}, \text{ so } dx = 2(u-1) du] \\ &= \lim_{t \rightarrow \infty} \left( [2\sqrt{t} - 2 \ln(\sqrt{t+1})] - (2 - 2 \ln 2) \right) \end{aligned}$$

Now  $2\sqrt{t} - 2 \ln(\sqrt{t+1}) = 2 \ln \left( \frac{e^{\sqrt{t}}}{\sqrt{t+1}} \right)$  and so

$$\lim_{t \rightarrow \infty} [2\sqrt{t} - 2 \ln(\sqrt{t+1})] = \infty \text{ (using l'Hospital's Rule)}$$

so both the integral and the original series diverge.

**A** [Click here for answers.](#)

9.  $f(x) = \frac{1}{x^2-1}$  is positive, continuous, and decreasing on  $[2, \infty)$ , so applying the Integral Test,

$$\begin{aligned} \int_2^{\infty} \frac{dx}{x^2-1} &= \int_2^{\infty} \left( \frac{-1/2}{x+1} + \frac{1/2}{x-1} \right) dx \\ &= \lim_{t \rightarrow \infty} \left[ \ln \left( \frac{x-1}{x+1} \right)^{1/2} \right]_2^t = \ln \sqrt{3} \Rightarrow \end{aligned}$$

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1} \text{ converges.}$$

10.  $f(x) = xe^{-x^2}$  is continuous and positive on  $[1, \infty)$ , and since  $f'(x) = e^{-x^2}(1-2x^2) < 0$  for  $x > 1$ ,  $f$  is decreasing as well. Thus, we can use the Integral Test:

$$\int_1^{\infty} xe^{-x^2} dx = \lim_{t \rightarrow \infty} \left[ -\frac{1}{2} e^{-x^2} \right]_1^t = 0 - \left( -\frac{1}{2} e^{-1} \right) = \frac{1}{2e}.$$

Since the integral converges, the series converges.

11.  $f(x) = \frac{x}{2^x}$  is positive and continuous on  $[1, \infty)$ , and since  $f'(x) = \frac{1-x \ln 2}{2^x} < 0$  when  $x > \frac{1}{\ln 2} \approx 1.44$ ,  $f$  is eventually decreasing, so we can apply the Integral Test. Integrating by parts, we get

$$\begin{aligned} \int_1^{\infty} \frac{x}{2^x} dx &= \lim_{t \rightarrow \infty} \left( -\frac{1}{\ln 2} \left[ \frac{x}{2^x} + \frac{1}{2^x \ln 2} \right]_1^t \right) \\ &= \frac{1}{2 \ln 2} + \frac{1}{2 (\ln 2)^2} \end{aligned}$$

since  $\lim_{t \rightarrow \infty} \frac{t}{2^t} = 0$  by l'Hospital's Rule, and so  $\sum_{n=1}^{\infty} \frac{n}{2^n}$  converges.

12.  $f(x) = \frac{1}{4x^2+1}$  is continuous, positive and decreasing on  $[1, \infty)$ , so applying the Integral Test,

$$\int_1^{\infty} \frac{dx}{4x^2+1} = \lim_{t \rightarrow \infty} \left[ \frac{\arctan 2x}{2} \right]_1^t = \frac{\pi}{4} - \frac{\arctan 2}{2} < \infty,$$

so the series converges.

13.  $f(x) = \frac{\arctan x}{1+x^2}$  is continuous and positive on  $[1, \infty)$ .

$$f'(x) = \frac{1-2x \arctan x}{(1+x^2)^2} < 0 \text{ for } x > 1, \text{ since}$$

$2x \arctan x \geq \frac{\pi}{2} > 1$  for  $x \geq 1$ . So  $f$  is decreasing and we can use the Integral Test.

$$\begin{aligned} \int_1^{\infty} \frac{\arctan x}{1+x^2} dx &= \lim_{t \rightarrow \infty} \left[ \frac{1}{2} (\arctan x)^2 \right]_1^t \\ &= \frac{(\pi/2)^2}{2} - \frac{(\pi/4)^2}{2} = \frac{3\pi^2}{32} \end{aligned}$$

so the series converges.

14.  $f(x) = \frac{\ln x}{x^2}$  is continuous and positive for  $x \geq 2$ , and  
 $f'(x) = \frac{1 - 2 \ln x}{x^3} < 0$  for  $x \geq 2$ , so  $f$  is decreasing.  
 $\int_2^{\infty} \frac{\ln x}{x^2} dx = \lim_{t \rightarrow \infty} \left[ -\frac{\ln x}{x} - \frac{1}{x} \right]_2^t$  (by parts)  $\stackrel{H}{=} 1$ . Thus,  
 $\sum_{n=1}^{\infty} \frac{\ln n}{n^2} = \sum_{n=2}^{\infty} \frac{\ln n}{n^2}$  converges by the Integral Test.

15.  $f(x) = \frac{1}{x^2 + 2x + 2}$  is continuous and positive on  $[1, \infty)$ ,  
and  $f'(x) = -\frac{2x + 2}{(x^2 + 2x + 2)^2} < 0$  for  $x \geq 1$ , so  $f$  is  
decreasing and we can use the Integral Test.  
 $\int_1^{\infty} \frac{1}{x^2 + 2x + 2} dx = \int_1^{\infty} \frac{1}{(x + 1)^2 + 1} dx$   
 $= \lim_{t \rightarrow \infty} [\arctan(x + 1)]_1^t$   
 $= \frac{\pi}{2} - \arctan 2$   
so the series converges as well.

16.  $\frac{1}{n^3 + n^2} < \frac{1}{n^3}$  since  $n^3 + n^2 > n^3$  for all  $n$ , and since  
 $\sum_{n=1}^{\infty} \frac{1}{n^3}$  is a convergent  $p$ -series ( $p = 3 > 1$ ),  $\sum_{n=1}^{\infty} \frac{1}{n^3 + n^2}$   
converges also by the Comparison Test.

17.  $\frac{3}{4^n + 5} < \frac{3}{4^n}$  and  $\sum_{n=1}^{\infty} \frac{3}{4^n}$  converges (geometric with  
 $|r| = \frac{1}{4} < 1$ ) so by the Comparison Test,  $\sum_{n=1}^{\infty} \frac{3}{4^n + 5}$   
converges also.

18.  $\frac{3}{n2^n} \leq \frac{3}{2^n}$ .  $\sum_{n=1}^{\infty} \frac{3}{2^n}$  is a geometric series with  
 $|r| = \frac{1}{2} < 1$ , and hence converges, so  $\sum_{n=1}^{\infty} \frac{3}{n2^n}$  converges  
also, by the Comparison Test.

19.  $\frac{1}{\sqrt{n} - 1} > \frac{1}{\sqrt{n}}$  and  $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$  diverges ( $p$ -series with  
 $p = \frac{1}{2} < 1$ ) so  $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n} - 1}$  diverges by the Comparison  
Test.

20.  $\frac{1 + 5^n}{4^n} > \frac{5^n}{4^n} = \left(\frac{5}{4}\right)^n$ .  $\sum_{n=0}^{\infty} \left(\frac{5}{4}\right)^n$  is a divergent  
geometric series ( $|r| = \frac{5}{4} > 1$ ) so  $\sum_{n=0}^{\infty} \frac{1 + 5^n}{4^n}$  diverges by  
the Comparison Test.

21.  $\frac{\sin^2 n}{n\sqrt{n}} \leq \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}$  and  $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$  converges  
( $p = \frac{3}{2} > 1$ ) so  $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n\sqrt{n}}$  converges by the Comparison  
Test.

22.  $\frac{3}{n(n+3)} < \frac{3}{n^2}$ .  $\sum_{n=1}^{\infty} \frac{3}{n^2} = 3 \sum_{n=1}^{\infty} \frac{1}{n^2}$  is a convergent  
 $p$ -series ( $p = 2 > 1$ ) so  $\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$  converges by the  
Comparison Test.

23.  $\frac{1}{\sqrt{n(n+1)(n+2)}} < \frac{1}{\sqrt{n \cdot n \cdot n}} = \frac{1}{n^{3/2}}$  and since  
 $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$  converges ( $p = \frac{3}{2} > 1$ ), so does  
 $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)(n+2)}}$  by the Comparison Test.

24. Use the Limit Comparison Test with  
 $a_n = \frac{1}{\sqrt[3]{n(n+1)(n+2)}}$  and  $b_n = \frac{1}{n}$ .  
 $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n}{\sqrt[3]{n(n+1)(n+2)}}$   
 $= \lim_{n \rightarrow \infty} \frac{1}{\sqrt[3]{1(1+1/n)(1+2/n)}}$   
 $= 1 > 0$   
so since  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges, so does  $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n(n+1)(n+2)}}$ .

25.  $\frac{n}{(n+1)2^n} < \frac{1}{2^n}$  and  $\sum_{n=1}^{\infty} \frac{1}{2^n}$  is a convergent geometric  
series ( $|r| = \frac{1}{2} < 1$ ), so  $\sum_{n=1}^{\infty} \frac{n}{(n+1)2^n}$  converges by the  
Comparison Test.

26.  $\frac{3 + \cos n}{3^n} \leq \frac{4}{3^n}$  since  $\cos n \leq 1$ .  $\sum_{n=1}^{\infty} \frac{4}{3^n}$  is a geometric  
series with  $|r| = \frac{1}{3} < 1$  so it converges, and so  
 $\sum_{n=1}^{\infty} \frac{3 + \cos n}{3^n}$  converges by the Comparison Test.

27.  $\frac{5n}{2n^2 - 5} > \frac{5n}{2n^2} = \frac{5}{2} \left(\frac{1}{n}\right)$  and since  $\frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n}$  diverges  
(harmonic series) so does  $\sum_{n=1}^{\infty} \frac{5n}{2n^2 - 5}$  by the Comparison  
Test.

28.  $\frac{n}{\sqrt{n^5 + 4}} < \frac{n}{\sqrt{n^5}} = \frac{1}{n^{3/2}}$ .  $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$  is a convergent  
 $p$ -series ( $p = \frac{3}{2} > 1$ ) so  $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^5 + 4}}$  converges by the  
Comparison Test.

29.  $\frac{\arctan n}{n^4} < \frac{\pi/2}{n^4}$  and  $\frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^4}$  converges ( $p = 4 > 1$ ) so

$\sum_{n=1}^{\infty} \frac{\arctan n}{n^4}$  converges by the Comparison Test.

30. Use the Limit Comparison Test with  $a_n = \frac{1}{n^2 - 4}$  and

$b_n = \frac{1}{n^2}$ :  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2 - 4} = 1 > 0$ . Since  $\sum_{n=3}^{\infty} b_n$

converges ( $p = 2 > 1$ ),  $\sum_{n=3}^{\infty} \frac{1}{n^2 - 4}$  also converges.

31. Let  $a_n = \frac{n^2 + 1}{n^4 + 1}$  and  $b_n = \frac{1}{n^2}$ . Then

$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n^4 + n^2}{n^4 + 1} = 1 > 0$ . Since  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  is a

convergent  $p$ -series ( $p = 2 > 1$ ), so is  $\sum_{n=1}^{\infty} \frac{n^2 + 1}{n^4 + 1}$  by the

Limit Comparison Test.

32. Let  $a_n = \frac{n+1}{n2^n}$  and  $b_n = \frac{1}{2^n}$ . Then

$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n+1}{n} = 1 > 0$ . Since  $\sum_{n=1}^{\infty} \frac{1}{2^n}$  is a

convergent geometric series ( $|r| = \frac{1}{2} < 1$ ),  $\sum_{n=1}^{\infty} \frac{n+1}{n2^n}$

converges by the Limit Comparison Test.

33. Use the Limit Comparison Test with  $a_n = \frac{n^2 - 3n}{\sqrt[3]{n^{10} - 4n^2}}$  and

$b_n = \frac{1}{n^{4/3}}$ .

$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n^{10/3} - 3n^{7/3}}{\sqrt[3]{n^{10} - 4n^2}} = \lim_{n \rightarrow \infty} \frac{1 - 3/n}{\sqrt[3]{1 - 4n^{-8}}}$   
 $= 1 > 0$

so since  $\sum_{n=1}^{\infty} b_n$  converges ( $p = \frac{4}{3} > 1$ ), so does

$\sum_{n=1}^{\infty} \frac{n^2 - 3n}{\sqrt[3]{n^{10} - 4n^2}}$ .