8.2 Series

A Click here for answers.

S Click here for solutions.

1–5 Find at least 10 partial sums of the series. Graph both the sequence of terms and the sequence of partial sums on the same screen. Does it appear that the series is convergent or divergent? If it is convergent, find the sum. If it is divergent, explain why.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{10}{3^n}$

2. $\displaystyle\sum_{n=1}^{\infty} \sin n$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$

4. $\displaystyle\sum_{n=4}^{\infty} \frac{3}{n(n-1)}$

5. $\displaystyle\sum_{n=1}^{\infty} \left(-\frac{2}{7}\right)^{n-1}$

6–33 Determine whether the series is convergent or divergent. If it is convergent, find its sum.

6. $4 + \frac{8}{5} + \frac{16}{25} + \frac{32}{125} + \cdots$

7. $1 - \frac{3}{2} + \frac{9}{4} - \frac{27}{8} + \cdots$

8. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$

9. $\frac{2}{3} - \frac{2}{9} + \frac{2}{27} - \frac{2}{81} + \cdots$

10. $-\frac{81}{100} + \frac{9}{10} - 1 + \frac{10}{9} - \cdots$

11. $\displaystyle\frac{1}{2^6} + \frac{1}{2^8} + \frac{1}{2^{10}} + \frac{1}{2^{12}} + \cdots$

12. $\frac{1}{36} + \frac{1}{30} + \frac{1}{25} + \frac{6}{125} + \cdots$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{e^{2n}}$

14. $\displaystyle\sum_{n=1}^{\infty} 3^{-n}8^{n+1}$

15. $\displaystyle\sum_{n=0}^{\infty} \frac{4^{n+1}}{5^n}$

16. $\displaystyle\sum_{n=1}^{\infty} \left(-\frac{3}{\pi}\right)^{n-1}$

17. $\displaystyle\sum_{n=1}^{\infty} 5\left(\frac{e}{3}\right)^n$

18. $\displaystyle\sum_{n=0}^{\infty} \frac{5^n}{8^n}$

19. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1}\frac{3^{2n}}{2^{3n+1}}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n}$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$

22. $\displaystyle\sum_{n=1}^{\infty} [2(0.1)^n + (0.2)^n]$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{1+n^2}}$

24. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{2^{n-1}} + \frac{2}{3^{n-1}}\right)$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5+2^{-n}}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{3(n+1)(n+2)}$

27. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(3n-2)(3n+1)}$

28. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n} + 2^n\right)$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$

30. $\displaystyle\sum_{n=1}^{\infty} \left[\sin\left(\frac{1}{n}\right) - \sin\left(\frac{1}{n+1}\right)\right]$

31. $\displaystyle\sum_{n=1}^{\infty} \ln\frac{n}{n+1}$

32. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$

33. $\displaystyle\sum_{n=2}^{\infty} \ln\frac{n^2-1}{n^2}$

34–38 Express the number as a ratio of integers.

34. $0.\overline{5} = 0.5555\ldots$

35. $0.\overline{15} = 0.15151515\ldots$

36. $0.\overline{307} = 0.307307307307\ldots$

37. $1.1\overline{23}$

38. $4.\overline{1570}$

39–43 Find the values of x for which the series converges. Find the sum of the series for those values of x.

39. $\displaystyle\sum_{n=0}^{\infty} 3^n x^n$

40. $\displaystyle\sum_{n=2}^{\infty} \frac{x^n}{5^n}$

41. $\displaystyle\sum_{n=0}^{\infty} 2^n \sin^n x$

42. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{x^n}$

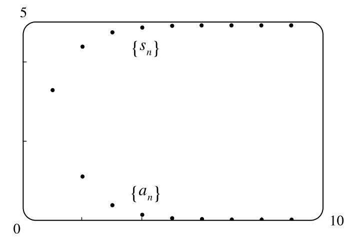
43. $\displaystyle\sum_{n=0}^{\infty} \tan^n x$

Answers

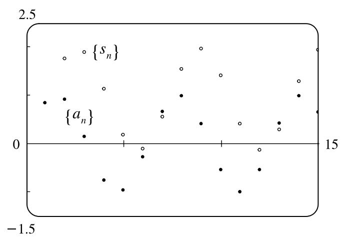
E Click here for exercises.

S Click here for solutions.

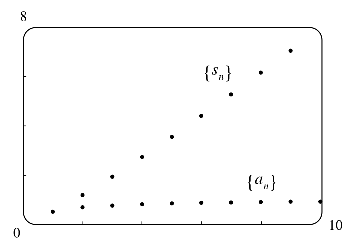
1. 3.33333, 4.44444, 4.81481, 4.93827, 4.97942, 4.99314, 4.99771, 4.99924, 4.99975, 4.99992
Convergent, sum = 5



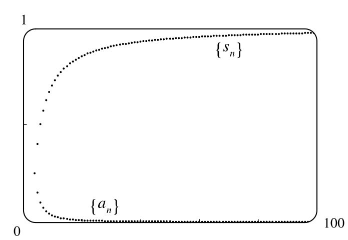
2. 0.8415, 1.7508, 1.8919, 1.1351, 0.1762, −0.1033, 0.5537, 1.5431, 1.9552, 1.4112
Divergent (terms do not approach 0)



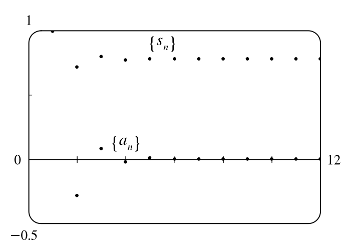
3. 0.50000, 1.16667, 1.91667, 2.71667, 3.55000, 4.40714, 5.28214, 6.17103, 7.07103, 7.98012
Divergent (terms do not approach 0)



4. 0.25000, 0.40000, 0.50000, 0.57143, 0.62500, 0.66667, 0.70000, 0.72727, 0.75000, 0.76923
Convergent, sum = 1



5. 1.000000, 0.714286, 0.795918, 0.772595, 0.779259, 0.777355, 0.777899, 0.777743, 0.777788, 0.777775
Convergent, sum = $\frac{7}{9}$



6. $\dfrac{20}{3}$

7. Divergent

8. $\dfrac{2}{3}$

9. $\dfrac{1}{2}$

10. Divergent

11. $\dfrac{1}{48}$

12. Divergent

13. $\dfrac{1}{e^2 - 1}$

14. Divergent

15. 20

16. $\dfrac{\pi}{\pi + 3}$

17. $\dfrac{5e}{3 - e}$

18. $\dfrac{8}{3}$

19. Divergent

20. Divergent

21. $\dfrac{3}{4}$

22. $\dfrac{17}{36}$

23. Divergent

24. 5

25. Divergent

26. Divergent

27. $\dfrac{1}{3}$

28. Divergent

29. $\dfrac{1}{2}$

30. $\sin 1$

31. Divergent

32. $\dfrac{1}{4}$

33. $\ln \frac{1}{2}$

34. $\dfrac{5}{9}$

35. $\dfrac{5}{33}$

36. $\dfrac{307}{999}$

37. $\dfrac{556}{495}$

38. $\dfrac{41{,}566}{9999}$

39. $-\frac{1}{3} < x < \frac{1}{3}$; $\dfrac{1}{1 - 3x}$

40. $-5 < x < 5$; $\dfrac{x^2}{25 - 5x}$

41. $n\pi - \frac{\pi}{6} < x < n\pi + \frac{\pi}{6}$ (n any integer); $\dfrac{1}{1 - 2\sin x}$

42. $|x| > 1$; $\dfrac{x}{x - 1}$

43. $n\pi - \frac{\pi}{4} < x < n\pi + \frac{\pi}{4}$ (n any integer); $\dfrac{1}{1 - \tan x}$

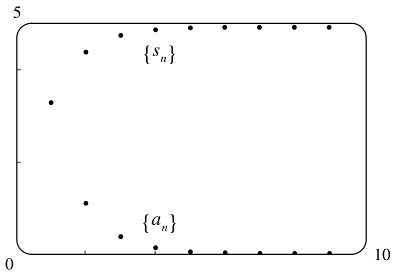
Solutions

E Click here for exercises.

A Click here for answers.

1.

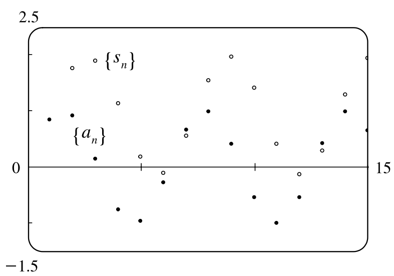
n	s_n
1	3.33333
2	4.44444
3	4.81481
4	4.93827
5	4.97942
6	4.99314
7	4.99771
8	4.99924
9	4.99975
10	4.99992
11	4.99997
12	4.99999



From the graph, it seems that the series converges. In fact, it is a geometric series with $a = \frac{10}{3}$ and $r = \frac{1}{3}$, so its sum is $\sum_{n=1}^{\infty} \frac{10}{3^n} = \frac{10/3}{1 - 1/3} = 5$. Note that the dot corresponding to $n = 1$ is part of both $\{a_n\}$ and $\{s_n\}$.

2.

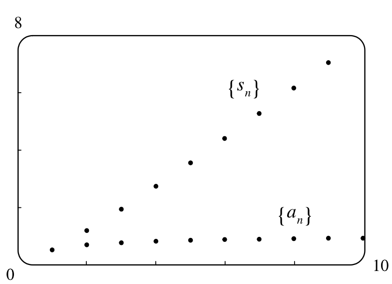
n	s_n
1	0.8415
2	1.7508
3	1.8919
4	1.1351
5	0.1762
6	−0.1033
7	0.5537
8	1.5431
9	1.9552
10	1.4112
11	0.4112
12	−0.1254



The series diverges, since its terms do not approach 0.

3.

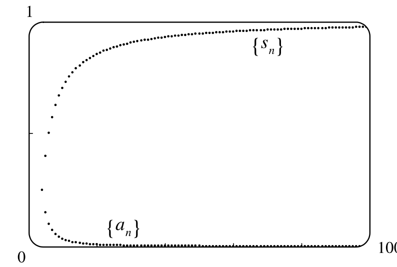
n	s_n
1	0.50000
2	1.16667
3	1.91667
4	2.71667
5	3.55000
6	4.40714
7	5.28214
8	6.17103
9	7.07103
10	7.98012



The series diverges,since its terms do not approach 0.

4.

n	s_n
4	0.25000
5	0.40000
6	0.50000
7	0.57143
8	0.62500
9	0.66667
10	0.70000
11	0.72727
12	0.75000
13	0.76923
...	...
99	0.96970
100	0.97000



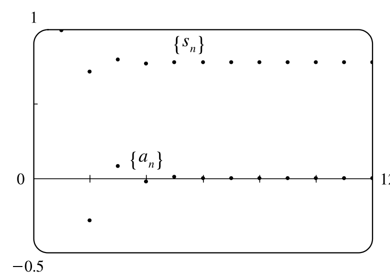
From the graph, it seems that the series converges to about 1. To find the sum, we proceed as in Example 6: since $\frac{3}{i(i-1)} = \frac{3}{i-1} - \frac{3}{i}$, the partial sums are

$$\begin{aligned} s_n &= \sum_{i=4}^{n} \left(\frac{3}{i-1} - \frac{3}{i} \right) \\ &= \left(\frac{3}{3} - \frac{3}{4} \right) + \left(\frac{3}{4} - \frac{3}{5} \right) + \cdots \\ &\quad + \left(\frac{3}{n-2} - \frac{3}{n-1} \right) + \left(\frac{3}{n-1} - \frac{3}{n} \right) \\ &= 1 - \frac{3}{n} \end{aligned}$$

and so the sum is $\lim_{n \to \infty} s_n = 1$.

5.

n	s_n
1	1.000000
2	0.714286
3	0.795918
4	0.772595
5	0.779259
6	0.777355
7	0.777899
8	0.777743
9	0.777788
10	0.777775
11	0.777779
12	0.777778



From the graph, it seems that the series converges to about 0.8. In fact, it is a geometric series with $a = 1$ and $r = -\frac{2}{7}$, so its sum is

$$\sum_{n=1}^{\infty} \left(-\frac{2}{7} \right)^{n-1} = \frac{1}{1 - (-2/7)} = \frac{7}{9}.$$

6. $4 + \frac{8}{5} + \frac{16}{25} + \frac{32}{125} + \cdots$ is a geometric series with $a = 4$ and $r = \frac{2}{5}$. Since $|r| = \frac{2}{5} < 1$, the series converges to $\frac{4}{1 - 2/5} = \frac{4}{3/5} = \frac{20}{3}$.

7. $1 - \frac{3}{2} + \frac{9}{4} - \frac{27}{8} + \cdots$ is a geometric series with $a = 1$ and $r = -\frac{3}{2}$. Since $|r| = \frac{3}{2} > 1$, the series diverges.

8. $a = 1$, $|r| = \left|-\frac{1}{2}\right| < 1$ so the series converges with sum $\dfrac{1}{1-(-1/2)} = \dfrac{2}{3}$.

9. $\sum\limits_{n=1}^{\infty} \frac{2}{3}\left(-\frac{1}{3}\right)^{n-1}$ is geometric with $a = \frac{2}{3}$, $r = -\frac{1}{3}$, so it converges to $\dfrac{2/3}{1-(-1/3)} = \dfrac{1}{2}$.

10. $a = -\frac{81}{100}$, $|r| = \left|-\frac{10}{9}\right| > 1$, so the series diverges.

11. $a = \dfrac{1}{2^6}$, $|r| = \frac{1}{4} < 1$, so the series converges with sum $\dfrac{1/2^6}{1-1/4} = \dfrac{1}{48}$.

12. $\sum\limits_{n=1}^{\infty} \frac{1}{36}\left(\frac{6}{5}\right)^{n-1}$ diverges since $r = \frac{6}{5} > 1$.

13. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{e^2}\right)^n \quad\Rightarrow\quad a = \frac{1}{e^2} = |r| < 1$, so the series converges to $\dfrac{1/e^2}{1-1/e^2} = \dfrac{1}{e^2-1}$.

14. For $\sum\limits_{n=1}^{\infty} 3^{-n}8^{n+1} = \sum\limits_{n=1}^{\infty} 8\left(\frac{8}{3}\right)^n$, $a = \frac{64}{3}$ and $r = \frac{8}{3} > 1$, so the series diverges.

15. $\sum\limits_{n=0}^{\infty} 4\left(\frac{4}{5}\right)^n \quad\Rightarrow\quad a = 4$, $|r| = \frac{4}{5} < 1$, so the series converges to $\dfrac{4}{1-4/5} = 20$.

16. $a = 1$, $|r| = \left|-\frac{3}{\pi}\right| < 1$, so the series converges to $\dfrac{1}{1-(-3/\pi)} = \dfrac{\pi}{\pi+3}$.

17. $a = \dfrac{5e}{3}$, $r = \dfrac{e}{3} < 1$, so the series converges to $\dfrac{5e/3}{1-e/3} = \dfrac{5e}{3-e}$.

18. $a = 1$, $r = \frac{5}{8} < 1$, so the series converges to $\dfrac{1}{1-5/8} = \dfrac{8}{3}$.

19. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{3^{2n}}{2^{3n+1}} = \sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)\left(-\frac{9}{8}\right)^n$, $|r| = \dfrac{9}{8} > 1$, so the series diverges.

20. This series diverges, since if it converged, so would $2\cdot\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n} = \sum_{n=1}^{\infty}\frac{1}{n}$ (by Theorem 8), which we know diverges (Example 7).

21. Converges.

$$s_n = \sum_{i=1}^{n}\frac{1}{i\,(i+2)} = \sum_{i=1}^{n}\left(\frac{1/2}{i} - \frac{1/2}{i+2}\right) \text{ (partial fractions}$$
$$= \frac{1}{2}\sum_{i=1}^{n}\left(\frac{1}{i} - \frac{1}{i+2}\right)$$

The latter sum is a telescoping series:

$$\left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\cdots + \left(\frac{1}{n-1}-\frac{1}{n+1}\right)+\left(\frac{1}{n}-\frac{1}{n+2}\right) = 1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}$$

Thus,

$$\sum_{n=1}^{\infty}\frac{1}{n\,(n+2)} = \frac{1}{2}\lim_{n\to\infty}\left(1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right) = \frac{1}{2}\left(1+\frac{1}{2}\right) = \frac{3}{4}$$

22. $\sum\limits_{n=1}^{\infty}\left[2\,(0.1)^n + (0.2)^n\right] = 2\sum\limits_{n=1}^{\infty}(0.1)^n + \sum\limits_{n=1}^{\infty}(0.2)^n$. These are convergent geometric series and so by Theorem 8, their sum is also convergent. $2\left(\frac{0.1}{1-0.1}\right) + \frac{0.2}{1-0.2} = \frac{2}{9}+\frac{1}{4} = \frac{17}{36}$

23. $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n}{\sqrt{1+n^2}} = \lim_{n\to\infty}\frac{1}{\sqrt{1+1/n^2}} = 1 \neq 0$, so the series diverges by the Test for Divergence.

24.
$$\sum_{n=1}^{\infty}\left(\frac{1}{2^{n-1}}+\frac{2}{3^{n-1}}\right) = \sum_{n=1}^{\infty}\frac{1}{2^{n-1}} + 2\sum_{n=1}^{\infty}\frac{1}{3^{n-1}} = \frac{1}{1-1/2} + 2\left(\frac{1}{1-1/3}\right) = 5$$

25. $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{5+2^{-n}} = \frac{1}{5} \neq 0$, so the series diverges by the Test for Divergence.

26.
$$\lim_{n\to\infty}\frac{n^2}{3\,(n+1)\,(n+2)} = \lim_{n\to\infty}\frac{1}{3\,(1+1/n)\,(1+2/n)} = \frac{1}{3} \neq 0$$
so the series diverges by the Test for Divergence.

27. Converges.

$$s_n = \sum_{i=1}^{n}\frac{1}{(3i-2)\,(3i+1)}$$
$$= \sum_{i=1}^{n}\left[\frac{1/3}{3i-2} - \frac{1/3}{3i+1}\right] \quad \text{(partial fractions)}$$
$$= \left[\frac{1}{3}\cdot 1 - \frac{1}{3}\cdot\frac{1}{4}\right] + \left[\frac{1}{3}\cdot\frac{1}{4} - \frac{1}{3}\cdot\frac{1}{7}\right] + \left[\frac{1}{3}\cdot\frac{1}{7} - \frac{1}{3}\cdot\frac{1}{10}\right] + \cdots + \left[\frac{1}{3}\cdot\frac{1}{3n-2} - \frac{1}{3}\cdot\frac{1}{3n+1}\right]$$
$$= \frac{1}{3} - \frac{1}{3\,(3n+1)} \quad \text{(telescoping series)}$$
$$\Rightarrow \lim_{n\to\infty} s_n = \frac{1}{3} \quad\Rightarrow\quad \sum_{n=1}^{\infty}\frac{1}{(3n-2)\,(3n+1)} = \frac{1}{3}$$

28. $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{1}{n}+2^n\right)$ does not exist, so the series diverges by the Test for Divergence.

29. $s_n = \sum_{i=1}^{n} \frac{1}{4i^2-1}$

$= \sum_{i=1}^{n} \left[\frac{1/2}{2i-1} - \frac{1/2}{2i+1}\right]$ (partial fractions)

$= \left(\frac{1}{2}\cdot 1 - \frac{1}{2}\cdot\frac{1}{3}\right) + \left(\frac{1}{2}\cdot\frac{1}{3} - \frac{1}{2}\cdot\frac{1}{5}\right) + \left(\frac{1}{2}\cdot\frac{1}{5} - \frac{1}{2}\cdot\frac{1}{7}\right) + \cdots + \left(\frac{1}{2}\cdot\frac{1}{2n-1} - \frac{1}{2}\cdot\frac{1}{2n+1}\right)$

$= \frac{1}{2} - \frac{1}{4n+2}$

so $\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \lim_{n\to\infty} s_n = \frac{1}{2}$.

30. Converges.

$s_n = \left(\sin 1 - \sin\frac{1}{2}\right) + \left(\sin\frac{1}{2} - \sin\frac{1}{3}\right) + \cdots + \left(\sin\frac{1}{n} - \sin\frac{1}{n+1}\right)$

$= \sin 1 - \sin\frac{1}{n+1}$, so

$\sum_{n=1}^{\infty}\left(\sin\frac{1}{n} - \sin\frac{1}{n+1}\right) = \lim_{n\to\infty} s_n = \sin 1 - \sin 0 = \sin 1$

31. $s_n = (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + (\ln 3 - \ln 4) + \cdots + [\ln n - \ln(n+1)]$

$= \ln 1 - \ln(n+1) = -\ln(n+1)$ (telescoping series).

Thus, $\lim_{n\to\infty} s_n = -\infty$, so the series is divergent.

32. $s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)(i+2)}$

$= \sum_{i=1}^{n}\left(\frac{1/2}{i} - \frac{1}{i+1} + \frac{1/2}{i+2}\right)$

$= \sum_{i=1}^{n}\left(\frac{1/2}{i} - \frac{1/2}{i+1}\right) + \sum_{i=1}^{n}\left(-\frac{1/2}{i+1} + \frac{1/2}{i+2}\right)$

both of which are clearly telescoping sums, so

$s_n = \left[\frac{1}{2} - \frac{1}{2(n+1)}\right] + \left[-\frac{1}{4} + \frac{1}{2(n+2)}\right]$

$= \frac{1}{4} - \frac{1}{2(n+1)} + \frac{1}{2(n+2)}$

Thus, $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \lim_{n\to\infty} s_n = \frac{1}{4}$.

33. Write $\ln\frac{n^2-1}{n^2} = \ln\frac{(n-1)(n+1)}{n\cdot n}$. Then

$s_n = \ln\frac{1\cdot 3}{2\cdot 2} + \ln\frac{2\cdot 4}{3\cdot 3} + \ln\frac{3\cdot 5}{4\cdot 4} + \cdots + \ln\frac{(n-2)n}{(n-1)(n-1)} + \ln\frac{(n-1)(n+1)}{n\cdot n}$

$= \ln\left(\frac{1\cdot 3}{2\cdot 2}\cdot\frac{2\cdot 4}{3\cdot 3}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot \cdots \cdot\frac{(n-2)n}{(n-1)(n-1)}\cdot\frac{(n-1)(n+1)}{n\cdot n}\right)$

$= \ln\frac{1}{2}\cdot\frac{n+1}{n}$

Therefore,

$\sum_{n=1}^{\infty}\ln\frac{n^2-1}{n^2} = \lim_{n\to\infty} s_n = \ln\frac{1}{2}\left(1+\frac{1}{n}\right) = \ln\frac{1}{2}$.

34. $0.\overline{5} = 0.5 + 0.05 + 0.005 + \cdots = \frac{0.5}{1-0.1} = \frac{5}{9}$

35. $0.\overline{15} = 0.15 + 0.0015 + 0.000015 + \cdots = \frac{0.15}{1-0.01}$

$= \frac{15}{99} = \frac{5}{33}$

36. $0.\overline{307} = 0.307 + 0.000307 + 0.000000307 + \cdots$

$= \frac{0.307}{1-0.001} = \frac{307}{999}$

37. $1.1\overline{23} = 1.1 + 0.023 + 0.00023 + 0.0000023 + \cdots$

$= 1.1 + \frac{0.023}{1-0.01} = \frac{11}{10} + \frac{23}{990} = \frac{556}{495}$

38. $4.\overline{1570} = 4 + 0.1570 + 0.00001570 + \cdots$

$= 4 + \frac{0.1570}{1-0.0001} = \frac{41{,}566}{9999}$

39. $\sum_{n=0}^{\infty}(3x)^n$ is geometric with $r = 3x$, so converges for $|3x| < 1 \Leftrightarrow -\frac{1}{3} < x < \frac{1}{3}$ to $\frac{1}{1-3x}$.

40. $\sum_{n=2}^{\infty}\left(\frac{x}{5}\right)^n$ is a geometric series with $r = \frac{x}{5}$, so converges whenever $\left|\frac{x}{5}\right| < 1 \Leftrightarrow -5 < x < 5$. The sum is $\frac{(x/5)^2}{1-x/5} = \frac{x^2}{25-5x}$.

41. $\sum_{n=0}^{\infty}(2\sin x)^n$ is geometric so converges whenever $|2\sin x| < 1 \Leftrightarrow -\frac{1}{2} < \sin x < \frac{1}{2} \Leftrightarrow n\pi - \frac{\pi}{6} < x < n\pi + \frac{\pi}{6}$, where the sum is $\frac{1}{1-2\sin x}$.

42. $\sum_{n=0}^{\infty}\left(\frac{1}{x}\right)^n$ is geometric with $r = \frac{1}{x}$, so it converges whenever $\left|\frac{1}{x}\right| < 1 \Leftrightarrow |x| > 1 \Leftrightarrow x > 1$ or $x < -1$, and the sum is $\frac{1}{1-1/x} = \frac{x}{x-1}$.

43. $\sum_{n=0}^{\infty}\tan^n x$ is geometric and converges when $|\tan x| < 1 \Leftrightarrow -1 < \tan x < 1 \Leftrightarrow n\pi - \frac{\pi}{4} < x < n\pi + \frac{\pi}{4}$ (n any integer). On these intervals the sum is $\frac{1}{1-\tan x}$.