8.1 Sequences

A Click here for answers.

S Click here for solutions.

1–8 List the first five terms of the sequence.

1. $a_n = \dfrac{n}{2n+1}$

2. $a_n = \dfrac{4n-3}{3n+4}$

3. $a_n = \dfrac{(-1)^{n-1}n}{2^n}$

4. $a_n = \left(-\dfrac{2}{3}\right)^n$

5. $\left\{\sin \dfrac{n\pi}{2}\right\}$

6. $a_1 = 1,\ a_{n+1} = \dfrac{1}{1+a_n}$

7. $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{n!}$

8. $\left\{\dfrac{(-7)^{n+1}}{n!}\right\}$

9–14 Find a formula for the general term a_n of the sequence, assuming that the pattern of the first few terms continues.

9. $\{1, 4, 7, 10, \ldots\}$

10. $\left\{\frac{3}{16}, \frac{4}{25}, \frac{5}{36}, \frac{6}{49}, \ldots\right\}$

11. $\left\{\frac{3}{2}, -\frac{9}{4}, \frac{27}{8}, -\frac{81}{16}, \ldots\right\}$

12. $\{-1, 2, -6, 24, \ldots\}$

13. $\left\{\frac{2}{3}, -\frac{3}{5}, \frac{4}{7}, -\frac{5}{9}, \ldots\right\}$

14. $\{0, 2, 0, 2, 0, 2, \ldots\}$

15–39 Determine whether the sequence converges or diverges. If it converges, find the limit.

15. $a_n = \dfrac{1}{4n^2}$

16. $a_n = 4\sqrt{n}$

17. $a_n = \dfrac{n^2-1}{n^2+1}$

18. $a_n = \dfrac{4n-3}{3n+4}$

19. $a_n = \dfrac{n^2}{n+1}$

20. $a_n = \dfrac{\sqrt[3]{n} + \sqrt[4]{n}}{\sqrt{n} + \sqrt[5]{n}}$

21. $a_n = (-1)^n \dfrac{n^2}{1+n^3}$

22. $\left\{\dfrac{\pi^n}{3^n}\right\}$

23. $a_n = \sin \dfrac{n\pi}{2}$

24. $a_n = 2 + \cos n\pi$

25. $\left\{\dfrac{3 + (-1)^n}{n^2}\right\}$

26. $\left\{\dfrac{n!}{(n+2)!}\right\}$

27. $\left\{\dfrac{\ln(n^2)}{n}\right\}$

28. $\left\{(-1)^n \sin \dfrac{1}{n}\right\}$

29. $\left\{\sqrt{n+2} - \sqrt{n}\right\}$

30. $\left\{\dfrac{\ln(2 + e^n)}{3n}\right\}$

31. $a_n = n2^{-n}$

32. $a_n = (1 + 3n)^{1/n}$

33. $a_n = n^{-1/n}$

34. $a_n = \left(\sqrt{n+1} - \sqrt{n}\right)\sqrt{n + \frac{1}{2}}$

35. $a_n = (-1)^{n-1} \dfrac{n^4}{1 + n^2 + n^3}$

36. $\left\{\arctan\left(\dfrac{2n}{2n+1}\right)\right\}$

37. $\left\{\dfrac{\sin n}{\sqrt{n}}\right\}$

38. $a_n = \dfrac{1}{n^2} + \dfrac{2}{n^2} + \cdots + \dfrac{n}{n^2}$

39. $a_n = \dfrac{n \cos n}{n^2 + 1}$

40–43 Determine whether the sequence is increasing, decreasing, or not monotonic. Is the sequence bounded?

40. $a_n = \dfrac{1}{3n+5}$

41. $a_n = \dfrac{n-2}{n+2}$

42. $a_n = \dfrac{3n+4}{2n+5}$

43. $a_n = \dfrac{\sqrt{n}}{n+2}$

Answers

E Click here for exercises.

S Click here for solutions.

1. $\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{11}$

2. $\frac{1}{7}, \frac{1}{2}, \frac{9}{13}, \frac{13}{16}, \frac{17}{19}$

3. $\frac{1}{2}, -\frac{1}{2}, \frac{3}{8}, -\frac{1}{4}, \frac{5}{32}$

4. $-\frac{2}{3}, \frac{4}{9}, \frac{-8}{27}, \frac{16}{81}, -\frac{32}{243}$

5. $1, 0, -1, 0, 1$

6. $1, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}$

7. $1, \frac{3}{2}, \frac{5}{2}, \frac{35}{8}, \frac{63}{8}$

8. $49, -\frac{343}{2}, \frac{2401}{6}, -\frac{16{,}807}{24}, \frac{117{,}649}{120}$

9. $a_n = 3n - 2$

10. $a_n = \frac{n+2}{(n+3)^2}$

11. $a_n = (-1)^{n+1} \left(\frac{3}{2}\right)^n$

12. $a_n = (-1)^n \; n!$

13. $a_n = (-1)^{n+1} \frac{n+1}{2n+1}$

14. $a_n = 1 - (-1)^{n-1}$ or $a_n = 1 + (-1)^n$

15. 0

16. Diverges

17. 1

18. $\frac{4}{3}$

19. Diverges

20. 0

21. 0

22. Diverges

23. Diverges

24. Diverges

25. 0

26. 0

27. 0

28. 0

29. 0

30. $\frac{1}{3}$

31. 0

32. 1

33. 1

34. $\frac{1}{2}$

35. Diverges

36. $\frac{\pi}{4}$

37. 0

38. $\frac{1}{2}$

39. 0

40. Decreasing; yes

41. Increasing; yes

42. Increasing; yes

43. Not monotonic; yes

Solutions

E Click here for exercises.

A Click here for answers.

1. $a_n = \dfrac{n}{2n+1}$, so the sequence is $\left\{\dfrac{1}{3}, \dfrac{2}{5}, \dfrac{3}{7}, \dfrac{4}{9}, \dfrac{5}{11}, \ldots\right\}$.

2. $a_n = \dfrac{4n-3}{3n+4}$, so the sequence is $\left\{\dfrac{1}{7}, \dfrac{1}{2}, \dfrac{9}{13}, \dfrac{13}{16}, \dfrac{17}{19}, \ldots\right\}$.

3. $a_n = \dfrac{(-1)^{n-1} n}{2^n}$, so the sequence is $\left\{\dfrac{1}{2}, -\dfrac{1}{2}, \dfrac{3}{8}, -\dfrac{1}{4}, \dfrac{5}{32}, \ldots\right\}$.

4. $a_n = \left(-\dfrac{2}{3}\right)^n$, so the sequence is $\left\{-\dfrac{2}{3}, \dfrac{4}{9}, \dfrac{-8}{27}, \dfrac{16}{81}, -\dfrac{32}{243}, \ldots\right\}$.

5. $a_n = \sin\dfrac{n\pi}{2}$, so the sequence is $\{1, 0, -1, 0, 1, \ldots\}$.

6. $a_1 = 1$, $a_{n+1} = \dfrac{1}{1+a_n}$, so the sequence is

$$\left\{1, \frac{1}{1+1}, \frac{1}{1+\frac{1}{2}}, \frac{1}{1+\frac{2}{3}}, \frac{1}{1+\frac{3}{5}}, \ldots\right\} = \left\{1, \frac{1}{2}, \frac{1}{\frac{3}{2}}, \frac{1}{\frac{5}{3}}, \frac{1}{\frac{8}{5}}\right\} = \left\{1, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \ldots\right\}.$$

7. $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{n!}$, so the sequence is $\left\{1, \dfrac{3}{2}, \dfrac{5}{2}, \dfrac{35}{8}, \dfrac{63}{8}, \ldots\right\}$.

8. $a_n = \dfrac{(-7)^{n+1}}{n!}$, so the sequence is $\left\{49, -\dfrac{343}{2}, \dfrac{2401}{6}, -\dfrac{16{,}807}{24}, \dfrac{117{,}649}{120}, \ldots\right\}$.

9. $a_n = 3n - 2$

10. $a_n = \dfrac{n+2}{(n+3)^2}$

11. $a_n = (-1)^{n+1}\left(\dfrac{3}{2}\right)^n$

12. $a_n = (-1)^n\, n!$

13. $a_n = (-1)^{n+1}\dfrac{n+1}{2n+1}$

14. $\{0, 2, 0, 2, 0, 2, \ldots\}$. 1 is halfway between 0 and 2, so we can think of alternately subtracting and adding 1 (from 1 and to 1) to obtain the given sequence: $a_n = 1 - (-1)^{n-1}$.

15. $\displaystyle\lim_{n\to\infty} \frac{1}{4n^2} = \frac{1}{4}\lim_{n\to\infty}\frac{1}{n^2} = \frac{1}{4}\cdot 0 = 0$. Convergent

16. $\{4\sqrt{n}\}$ clearly diverges since $\sqrt{n} \to \infty$ as $n \to \infty$.

17. $\displaystyle\lim_{n\to\infty}\frac{n^2-1}{n^2+1} = \lim_{n\to\infty}\frac{1-1/n^2}{1+1/n^2} = 1$. Convergent

18. $\displaystyle\lim_{n\to\infty}\frac{4n-3}{3n+4} = \lim_{n\to\infty}\frac{4-3/n}{3+4/n} = \frac{4}{3}$. Convergent

19. $\{a_n\}$ diverges since $\dfrac{n^2}{n+1} = \dfrac{n}{1+1/n} \to \infty$ as $n \to \infty$.

20. $\displaystyle\lim_{n\to\infty}\frac{n^{1/3}+n^{1/4}}{n^{1/2}+n^{1/5}} = \lim_{n\to\infty}\frac{1/n^{1/6}+1/n^{1/4}}{1+1/n^{3/10}} = \frac{0}{1} = 0$ so the sequence converges.

21. $\displaystyle\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\frac{n^2}{1+n^3} = \lim_{n\to\infty}\frac{1/n}{(1/n^3)+1} = 0$, so by Theorem 6, $\displaystyle\lim_{n\to\infty}(-1)^n\left(\frac{n^2}{1+n^3}\right) = 0$.

22. $a_n = \left(\frac{\pi}{3}\right)^n$, so $\{a_n\}$ diverges by Equation 8 with $r = \frac{\pi}{3} > 1$.

23. $\{a_n\} = \{1, 0, -1, 0, 1, 0, -1, \ldots\}$. This sequence oscillates among 1, 0, and -1, so the sequence diverges.

24. $a_n = 2 + \cos n\pi$, so

$$\begin{aligned}\{a_n\} &= \{2+\cos\pi, 2+\cos 2\pi, 2+\cos 3\pi, 2+\cos 4\pi, \ldots\}\\ &= \{2-1, 2+1, 2-1, 2+1, \ldots\}\\ &= \{1, 3, 1, 3, \ldots\}\end{aligned}$$

This sequence oscillates between 1 and 3, so it diverges.

25. $0 < \dfrac{3+(-1)^n}{n^2} \le \dfrac{4}{n^2}$ and $\displaystyle\lim_{n\to\infty}\frac{4}{n^2} = 0$, so $\left\{\dfrac{3+(-1)^n}{n^2}\right\}$ converges to 0 by the Squeeze Theorem.

26.
$$\begin{aligned}\lim_{n\to\infty}\frac{n!}{(n+2)!} &= \lim_{n\to\infty}\frac{1\cdot 2\cdot 3\cdot\cdots\cdot n}{1\cdot 2\cdot 3\cdot\cdots\cdot n\,(n+1)\,(n+2)}\\ &= \lim_{n\to\infty}\frac{1}{(n+2)\,(n+1)} = 0\end{aligned}$$

Convergent

27. $\displaystyle\lim_{x\to\infty}\frac{\ln(x^2)}{x} = \lim_{x\to\infty}\frac{2\ln x}{x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{2/x}{1} = 0$, so by Theorem 3, $\left\{\dfrac{\ln(n^2)}{n}\right\}$ converges to 0.

28. $\displaystyle\lim_{n\to\infty}\sin\left(\frac{1}{n}\right) = \sin 0 = 0$ since $\dfrac{1}{n} \to 0$ as $n \to \infty$, so by Theorem 6, $\left\{(-1)^n\sin\left(\dfrac{1}{n}\right)\right\}$ converges to 0.

29. $b_n = \sqrt{n+2} - \sqrt{n} = \left(\sqrt{n+2}-\sqrt{n}\right)\dfrac{\sqrt{n+2}+\sqrt{n}}{\sqrt{n+2}+\sqrt{n}}$

$$= \frac{2}{\sqrt{n+2}+\sqrt{n}} < \frac{2}{2\sqrt{n}} = \frac{1}{\sqrt{n}} \to 0 \text{ as } n \to \infty.$$

So by the Squeeze Theorem with $a_n = 0$ and $c_n = 1/\sqrt{n}$, $\left\{\sqrt{n+2}-\sqrt{n}\right\}$ converges to 0.

30. $\lim_{x\to\infty} \frac{\ln(2+e^x)}{3x} \overset{H}{=} \lim_{x\to\infty} \frac{e^x/(2+e^x)}{3}$

$= \lim_{x\to\infty} \frac{1}{6e^{-x}+3} = \frac{1}{3}$

so by Theorem 3, $\lim_{n\to\infty} \frac{\ln(2+e^n)}{3n} = \frac{1}{3}$. Convergent

31. $\lim_{x\to\infty} \frac{x}{2^x} \overset{H}{=} \lim_{x\to\infty} \frac{1}{(\ln 2)\, 2^x} = 0$, so by Theorem 3, $\{n2^{-n}\}$ converges to 0.

32. $y = (1+3x)^{1/x} \Rightarrow \ln(y) = \frac{1}{x}\ln(1+3x) \Rightarrow$

$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(1+3x)}{x} \overset{H}{=} \lim_{x\to\infty} \frac{3/(1+3x)}{1} = 0$

$\Rightarrow \lim_{x\to\infty} y = e^0 = 1$, so by Theorem 3, $\left\{(1+3n)^{1/n}\right\}$ converges to 1.

33. Let $y = x^{-1/x}$. Then $\ln y = -\frac{\ln x}{x}$ and

$\lim_{x\to\infty} (\ln y) \overset{H}{=} \lim_{x\to\infty} \left(-\frac{1/x}{1}\right) = 0$, so $\lim_{x\to\infty} y = e^0 = 1$,

and so $\{a_n\}$ converges to 1.

34. $a_n = \left(\sqrt{n+1} - \sqrt{n}\right)\sqrt{n+\frac{1}{2}}$

$= \left(\sqrt{n+1} - \sqrt{n}\right)\left(\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right)\sqrt{n+\frac{1}{2}}$

$= \frac{\sqrt{n+1/2}}{\sqrt{n+1}+\sqrt{n}} = \frac{\sqrt{1+1/(2n)}}{\sqrt{1+1/n}+1} \to \frac{1}{2}$ as $n\to\infty$.

Convergent

35. $|a_n| = \frac{n}{1/n^3 + 1/n + 1} \to \infty$ as $n \to \infty$, so $\{a_n\}$ diverges.

36. $\lim_{n\to\infty} \frac{2n}{2n+1} = \lim_{n\to\infty} \frac{2}{2+1/n} = 1$, so

$\lim_{n\to\infty} \arctan\left(\frac{2n}{2n+1}\right) = \arctan 1 = \frac{\pi}{4}$. Convergent.

37. $0 < \frac{|\sin n|}{\sqrt{n}} \le \frac{1}{\sqrt{n}} \to 0$ as $n\to\infty$, so by the Squeeze Theorem and Theorem 6, $\left\{\frac{\sin n}{\sqrt{n}}\right\}$ converges to 0.

38. The series converges, since

$a_n = \frac{1+2+3+\cdots+n}{n^2}$

$= \frac{n(n+1)/2}{n^2}$ [sum of the first n positive integers]

$= \frac{n+1}{2n} = \frac{1+1/n}{2} \to \frac{1}{2}$ as $n\to\infty$.

39. $0 \le |a_n| = \frac{n|\cos n|}{n^2+1} \le \frac{n}{n^2+1} = \frac{1}{n+1/n} \to 0$ as $n\to\infty$, so by the Squeeze Theorem and Theorem 6, $\{a_n\}$ converges to 0.

40. $3(n+1)+5 > 3n+5$ so $\frac{1}{3(n+1)+5} < \frac{1}{3n+5} \Leftrightarrow$ $a_{n+1} < a_n$ so $\{a_n\}$ is decreasing. The sequence is bounded because $a_n = \frac{1}{3n+5} > 0$ for $n \ge 1$.

41. $\left\{\frac{n-2}{n+2}\right\}$ is increasing since

$a_n < a_{n+1} \Leftrightarrow \frac{n-2}{n+2} < \frac{(n+1)-2}{(n+1)+2}$

$\Leftrightarrow (n-2)(n+3) < (n+2)(n-1) \Leftrightarrow$

$n^2+n-6 < n^2+n-2 \Leftrightarrow -6 < -2$, which is of course true. The sequence is bounded because

$\frac{n-2}{n+2} < \frac{n+2}{n+2} = 1$ for $n \ge 1$.

42. $\left\{\frac{3n+4}{2n+5}\right\}$ is increasing since $a_{n+1} \ge a_n$

$\Leftrightarrow \frac{3(n+1)+4}{2(n+1)+5} \ge \frac{3n+4}{2n+5} \Leftrightarrow$

$(3n+7)(2n+5) \ge (3n+4)(2n+7) \Leftrightarrow$

$6n^2+29n+35 \ge 6n^2+29n+28 \Leftrightarrow 35 \ge 28$. The sequence is bounded because $a_n = \frac{3n+4}{2n+5} < \frac{4n+10}{2n+5} = 2$ for $n \ge 1$.

43. $a_n = \frac{\sqrt{n}}{n+2}$ defines a sequence that is neither increasing nor decreasing since $a_1 < a_2$ and $a_2 > a_3$. ($a_1 = \frac{1}{3} = 0.\overline{3}$, $a_2 = \frac{\sqrt{2}}{4} \approx 0.354$, and $a_3 = \frac{\sqrt{3}}{5} \approx 0.346$.) But the sequence $\{a_n \mid n \ge 2\}$ obtained by omitting the first term a_1 *is* decreasing. To see this, note that if $f(x) = \frac{\sqrt{x}}{x+2}$ for $x \ge 0$, then

$$f'(x) = \frac{\frac{x+2}{2\sqrt{x}} - \sqrt{x}}{(x+2)^2} = \frac{(x+2)-2x}{2\sqrt{x}\,(x+2)^2}$$

$$= \frac{2-x}{2\sqrt{x}\,(x+2)^2} \le 0 \text{ for } x \ge 2.$$

The sequence is bounded since $a_n > 0$ for all $n \ge 1$ and $a_n \le a_2 = \frac{\sqrt{2}}{4}$ for all $n \ge 1$.