7.4 Exponential Growth and Decay

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S Click here for solutions.

1. A bacteria culture starts with 4000 bacteria and the population triples every half-hour.
(a) Find an expression for the number of bacteria after t hours.
(b) Find the number of bacteria after 20 min.
(c) When will the population reach 20,000?

2. A bacteria culture grows with constant relative growth rate. The count was 400 after 2 hours and 25,600 after 6 hours.
(a) What was the initial population of the culture?
(b) Find an expression for the population after t hours.
(c) In what period of time does the population double?
(d) When will the population reach 100,000?

3. Polonium-210 has a half-life of 140 days.
(a) If a sample has a mass of 200 mg, find a formula for the mass that remains after t days.
(b) Find the mass after 100 days.
(c) When will the mass be reduced to 10 mg?
(d) Sketch the graph of the mass function.

4. Polonium-214 has a very short half-life of 1.4×10^{-4} s.
(a) If a sample has a mass of 50 mg, find a formula for the mass that remains after t seconds.
(b) Find the mass that remains after a hundredth of a second.
(c) How long would it take for the mass to decay to 40 mg?

5. On a hot day a thermometer is taken outside from an air-conditioned room where the temperature is 21°C. After one minute it reads 27°C and after 2 minutes it reads 30°C.
(a) What is the outdoor temperature?
(b) Sketch the graph of the temperature function.

Answers

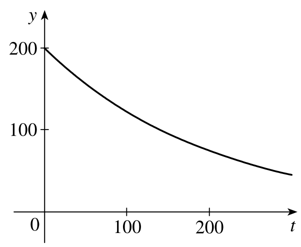
E Click here for exercises.

S Click here for solutions.

1. (a) $y(t) = 4000 \cdot 9^t$
(b) 8320
(c) ≈ 44 min

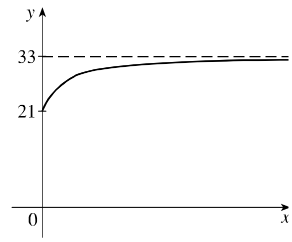
2. (a) 50
(b) $y(t) = 50 \cdot 8^{t/2}$
(c) 40 min.
(d) ≈ 7.3 h

3. (a) $y(t) = 200 \cdot 2^{-t/140}$
(b) ≈ 121.9 mg
(c) ≈ 605 days
(d)



4. (a) $y(t) = 50 \cdot 2^{-t/0.00014}$
(b) $\approx 1.57 \times 10^{-20}$ mg
(c) $\approx 4.5 \times 10^{-5}$ s

5. (a) $33°$
(b)



Solutions

E Click here for exercises.

A Click here for answers.

1. (a) By (2), $y(t) = y(0)\, e^{kt} = 4000e^{kt} \quad \Rightarrow$

$y\left(\frac{1}{2}\right) = 4000e^{k/2} = 12{,}000 \quad \Rightarrow \quad e^{k/2} = 3 \quad \Rightarrow$

$k/2 = \ln 3 \quad \Rightarrow \quad k = 2\ln 3$, so

$y(t) = 4000e^{(2\ln 3)t} = 4000 \cdot 9^t$.

(b) $y\left(\frac{1}{3}\right) = 4000 \cdot 9^{1/3} \approx 8320$

(c) $4000 \cdot 9^t = 20{,}000 \quad \Rightarrow \quad 9^t = 5 \quad \Rightarrow \quad t\ln 9 = \ln 5$

$\Rightarrow \quad t = (\ln 5) / (\ln 9) \approx 0.73 \text{ h} \approx 44 \text{ min}$

2. (a) $y(t) = y(0)\, e^{kt} \quad \Rightarrow \quad y(2) = y(0)\, e^{2k} = 400$,

$y(6) = y(0)\, e^{6k} = 25{,}600$. Dividing these equations,

we get $e^{6k}/e^{2k} = 25{,}600/400 \quad \Rightarrow \quad e^{4k} = 64 \quad \Rightarrow$

$4k = \ln 64 = 6\ln 2 \quad \Rightarrow \quad k = \frac{3}{2}\ln 2 = \frac{1}{2}\ln 8$. Thus,

$y(0) = 400/e^{2k} = 400/e^{\ln 8} = \frac{400}{8} = 50$.

(b) $y(t) = y(0)\, e^{kt} = 50e^{(\ln 8)t/2}$ or $y = 50 \cdot 8^{t/2}$

(c) $y(t) = 50e^{(3\ln 2)t/2} = 100 \quad \Leftrightarrow \quad e^{(3\ln 2)t/2} = 2 \quad \Leftrightarrow$

$(3\ln 2)\, t/2 = \ln 2 \quad \Leftrightarrow \quad t = 2/3 \text{ h} = 40 \text{ min}$

(d) $50e^{(\ln 8)t/2} = 100{,}000 \quad \Leftrightarrow \quad e^{(\ln 8)t/2} = 2000$

$\Leftrightarrow \quad (\ln 8)\, t/2 = \ln 2000 \quad \Leftrightarrow$

$t = (2\ln 2000) / \ln 8 \approx 7.3$ h.

3. (a) The mass remaining after t days is

$y(t) = y(0)\, e^{kt} = 200e^{kt}$. Since the half-life is

140 days, $y(140) = 200e^{140k} = 100 \quad \Rightarrow \quad e^{140k} = \frac{1}{2}$

$\Rightarrow \quad 140k = \ln\frac{1}{2} \quad \Rightarrow \quad k = -(\ln 2)/140$, so

$y(t) = 200e^{-(\ln 2)t/140} = 200 \cdot 2^{-t/140}$.

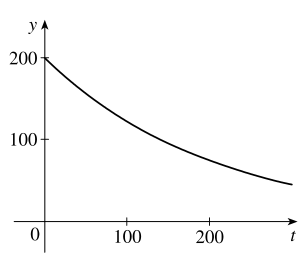
(b) $y(100) = 200 \cdot 2^{-100/140} \approx 121.9$ mg

(c) $200e^{-(\ln 2)t/140} = 10 \quad \Leftrightarrow$

$-\ln 2\frac{t}{140} = \ln\frac{1}{20} = -\ln 20 \quad \Leftrightarrow$

$t = 140\frac{\ln 20}{\ln 2} \approx 605$ days

(d)



4. (a) If $y(t)$ is the mass remaining after t days,

then $y(t) = y(0)\, e^{kt} = 50e^{kt}$.

$y(0.00014) = 50e^{0.00014k} = 25 \quad \Rightarrow$

$e^{0.00014k} = \frac{1}{2} \quad \Rightarrow \quad k = -(\ln 2)/0.00014 \quad \Rightarrow$

$y(t) = 50e^{-(\ln 2)t/0.00014} = 50 \cdot 2^{-t/0.00014}$

(b) $y(0.01) = 50 \cdot 2^{-0.01/0.00014} \approx 1.57 \times 10^{-20}$ mg

(c) $50e^{-(\ln 2)t/0.00014} = 40 \quad \Rightarrow$

$-(\ln 2)\, t/0.00014 = \ln 0.8 \quad \Rightarrow$

$t = -0.00014\dfrac{\ln 0.8}{\ln 2} \approx 4.5 \times 10^{-5}$ s

5. (a) Let $y(t)$ = temperature after t minutes. Newton's Law of Cooling implies that $\dfrac{dy}{dt} = k(y - a)$ where a is the surrounding temperature. Let $u(t) = y(t) - a$. Then $\dfrac{du}{dt} = ku$, so $u(t) = u(0)\, e^{kt} = (21 - a)\, e^{kt}$ and so $y(t) = a + (21 - a)\, e^{kt}$. Using the knowledge of $y(t)$ at $t = 1$ and $t = 2$ we have: $27 = y(1) = a + (21 - a)\, e^k$ and $30 = y(2) = a + (21 - a)\, e^{2k}$. Rearranged, these become $27 - a = (21 - a)\, e^k$ and $30 - a = (21 - a)\, e^{2k}$. To determine a we must eliminate k. To do so, we divide the square of the first equation by the second and get

$$\frac{(27 - a)^2}{30 - a} = \frac{(21 - a)^2 e^{2k}}{(21 - a)\, e^{2k}} \quad \Rightarrow \quad \frac{(27 - a)^2}{30 - a} = 21 - a$$

$\Rightarrow \quad (27 - a)^2 = (30 - a)(21 - a) \quad \Leftrightarrow$

$729 - 54a + a^2 = 630 - 51a + a^2 \quad \Rightarrow \quad -3a = -99$

$\Rightarrow \quad a = 33$. Hence, the outdoor temperature is $33°$.

(b)