7.3 Separable Equations

A Click here for answers.

S Click here for solutions.

1–8 Solve the differential equation.

1. $\dfrac{dy}{dx} = y^2$

2. $yy' = x$

3. $y' = xy$

4. $\dfrac{dy}{dx} = \dfrac{x + \sin x}{3y^2}$

5. $x^2y' + y = 0$

6. $y' = \dfrac{\ln x}{xy + xy^3}$

7. $\dfrac{du}{dt} = e^{u+2t}$

8. $\dfrac{dx}{dt} = 1 + t - x - tx$

9–14 Find the solution of the differential equation that satisfies the given initial condition.

9. $\dfrac{dy}{dx} = \dfrac{1 + x}{xy}, \quad x > 0, \quad y(1) = -4$

10. $xe^{-t}\dfrac{dx}{dt} = t, \quad x(0) = 1$

11. $x + 2y\sqrt{x^2 + 1}\,\dfrac{dy}{dx} = 0, \quad y(0) = 1$

12. $e^y y' = \dfrac{3x^2}{1 + y}, \quad y(2) = 0$

13. $\dfrac{du}{dt} = \dfrac{2t + 1}{2(u - 1)}, \quad u(0) = -1$

14. $\dfrac{dy}{dt} = \dfrac{ty + 3t}{t^2 + 1}, \quad y(2) = 2$

15. Solve the initial-value problem $y' = y \sin x$, $y(0) = 1$, and graph the solution.

16. Find a function f such that $f'(x) = x^3 f(x)$ and $f(0) = 1$.

17. Find a function g such that $g'(x) = g(x)(1 + g(x))$ and $g(0) = 1$.

Answers

E Click here for exercises.

S Click here for solutions.

1. $y = \dfrac{-1}{x+C}, y = 0$

2. $x^2 - y^2 = C$

3. $y = Ce^{x^2/2}$

4. $y = \sqrt[3]{\frac{1}{2}x^2 - \cos x + C}$

5. $y = Ce^{1/x}$

6. $y^2 + 1 = \sqrt{2(\ln x)^2 + C}$

7. $u = -\ln\left(C - \frac{1}{2}e^{2t}\right)$

8. $x = 1 + Ce^{-\left(t^2/2+t\right)}$

9. $y^2 = 2\ln x + 2x + 14$

10. $x = \sqrt{2(t-1)e^t + 3}$

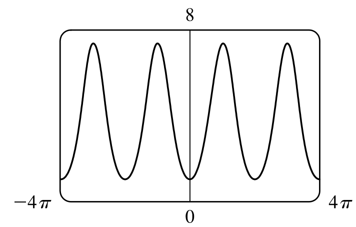
11. $y^2 = 2 - \sqrt{x^2+1}$

12. $ye^y = x^3 - 8$

13. $u = 1 - \sqrt{t^2 + t + 4}$

14. $y = -3 + \sqrt{5t^2 + 5}$

15. $y = e^{1-\cos x}$



16. $f(x) = e^{x^4/4}$

17. $g(x) = \dfrac{e^x}{2 - e^x}$

Solutions

E Click here for exercises.

A Click here for answers.

1. $\dfrac{dy}{dx} = y^2 \;\Rightarrow\; \dfrac{dy}{y^2} = dx \; (y \neq 0) \;\Rightarrow\; \displaystyle\int \frac{dy}{y^2} = \int dx$
$\Rightarrow\; -\dfrac{1}{y} = x + C \;\Rightarrow\; -y = \dfrac{1}{x+C} \;\Rightarrow\; y = \dfrac{-1}{x+C}$,
and $y = 0$ is also a solution.

2. $yy' = x \;\Rightarrow\; \int y\,dy = \int x\,dx \;\Rightarrow$
$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C_1 \;\Rightarrow\; y^2 = x^2 + 2C_1 \;\Rightarrow$
$x^2 - y^2 = C$ (where $C = -2C_1$). This represents a family of hyperbolas.

3. $y' = xy \;\Rightarrow\; \displaystyle\int \frac{dy}{y} = \int x\,dx \quad (y \neq 0) \;\Rightarrow$
$\ln|y| = \dfrac{x^2}{2} + C \;\Rightarrow\; |y| = e^C e^{x^2/2} \;\Rightarrow\; y = Ke^{x^2/2}$,
where $K = \pm e^C$ is a constant. (In our derivation, K was nonzero, but we can restore the excluded case $y = 0$ by allowing K to be zero.)

4. $\dfrac{dy}{dx} = \dfrac{x + \sin x}{3y^2} \;\Rightarrow\; \int 3y^2\,dy = \int (x + \sin x)\,dx \;\Rightarrow$
$y^3 = \dfrac{x^2}{2} - \cos x + C \;\Rightarrow\; y = \sqrt[3]{\frac{1}{2}x^2 - \cos x + C}$

5. $x^2y' + y = 0 \;\Rightarrow\; \dfrac{dy}{dx} = -\dfrac{y}{x^2} \;\Rightarrow$
$\displaystyle\int \frac{dy}{y} = \int \frac{-dx}{x^2} \quad (y \neq 0) \;\Rightarrow\; \ln|y| = \frac{1}{x} + K \;\Rightarrow$
$|y| = e^K e^{1/x} \;\Rightarrow\; y = Ce^{1/x}$, where now we allow C to be any constant.

6. $y' = \dfrac{\ln x}{xy + xy^3} = \dfrac{\ln x}{x(y + y^3)} \;\Rightarrow$
$\displaystyle\int (y + y^3)\,dy = \int \frac{\ln x}{x}\,dx \;\Rightarrow$
$\dfrac{y^2}{2} + \dfrac{y^4}{4} = \frac{1}{2}(\ln x)^2 + C_1 \;\Rightarrow$
$y^4 + 2y^2 = 2(\ln x)^2 + 2C_1 \;\Rightarrow$
$(y^2 + 1)^2 = 2(\ln x)^2 + K \quad$ (where $K = 2C_1 + 1$)
$\Rightarrow\; y^2 + 1 = \sqrt{2(\ln x)^2 + K}$

7. $\dfrac{du}{dt} = e^{u+2t} = e^u e^{2t} \;\Rightarrow\; \int e^{-u}du = \int e^{2t}dt \;\Rightarrow$
$-e^{-u} = \frac{1}{2}e^{2t} + C_1 \;\Rightarrow\; e^{-u} = -\frac{1}{2}e^{2t} + C$ (where $C = -C_1$ and the right-hand side is positive, since $e^{-u} > 0$)
$\Rightarrow\; -u = \ln\left(C - \frac{1}{2}e^{2t}\right) \;\Rightarrow\; u = -\ln\left(C - \frac{1}{2}e^{2t}\right)$

8. $\dfrac{dx}{dt} = 1 + t - x - tx = (1+t)(1-x) \;\Rightarrow$
$\displaystyle\int \frac{dx}{1-x} = \int (1+t)\,dt \quad (x \neq 1) \;\Rightarrow$
$-\ln|1-x| = \frac{1}{2}t^2 + t + C \;\Rightarrow\; |1-x| = e^{-(t^2/2+t+C)}$
$\Rightarrow\; 1 - x = \pm e^{-(t^2/2+t+C)} \;\Rightarrow$
$x = 1 + Ae^{-(t^2/2+t)}$ (where $A = \pm e^C$ or 0)

9. $\dfrac{dy}{dx} = \dfrac{1+x}{xy}$, $x > 0$, $y(1) = -4$.
$\displaystyle\int y\,dy = \int \frac{1+x}{x}\,dx = \int \left(\frac{1}{x} + 1\right)dx \;\Rightarrow$
$\frac{1}{2}y^2 = \ln|x| + x + C = \ln x + x + C$ (since $x > 0$).
$y(1) = -4 \;\Rightarrow\; \frac{(-4)^2}{2} = \ln 1 + 1 + C \;\Rightarrow$
$8 = 0 + 1 + C \;\Rightarrow\; C = 7$, so $y^2 = 2\ln x + 2x + 14$.

10. $xe^{-t}\dfrac{dx}{dt} = t$, $x(0) = 1$. $\int x\,dx = \int te^t\,dt \;\Rightarrow$
$\frac{1}{2}x^2 = (t-1)e^t + C$ [integration by parts or Formula 96].
$x(0) = 1$, so $\frac{1}{2} = (0-1)e^0 + C$ and $C = \frac{3}{2}$. Thus,
$x^2 = 2(t-1)e^t + 3 \;\Rightarrow\; x = \sqrt{2(t-1)e^t + 3}$ [use the positive square root since $x(0) = +1$].

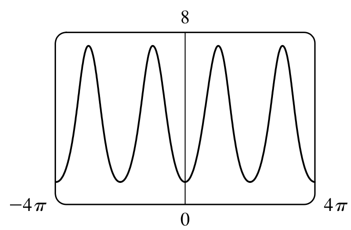
11. $x + 2y\sqrt{x^2+1}\,\dfrac{dy}{dx} = 0 \;\Rightarrow\; x\,dx + 2y\sqrt{x^2+1}\,dy = 0$,
$y(0) = 1$. $\displaystyle\int 2y\,dy = -\int \frac{x\,dx}{\sqrt{x^2+1}} \;\Rightarrow$
$y^2 = -\sqrt{x^2+1} + C$. $y(0) = 1 \;\Rightarrow\; 1 = -1 + C \;\Rightarrow$
$C = 2$, so $y^2 = 2 - \sqrt{x^2+1}$.

12. $e^y y' = \dfrac{3x^2}{1+y}$, $y(2) = 0$. $\int e^y(1+y)\,dy = \int 3x^2\,dx \;\Rightarrow$
$ye^y = x^3 + C$. $y(2) = 0$, so $0 = 2^3 + C$ and $C = -8$.
Thus $ye^y = x^3 - 8$.

13. $\dfrac{du}{dt} = \dfrac{2t+1}{2(u-1)}$, $u(0) = -1$.
$\int 2(u-1)\,du = \int (2t+1)\,dt \;\Rightarrow$
$u^2 - 2u = t^2 + t + C$. $u(0) = -1$ so
$(-1)^2 - 2(-1) = 0^2 + 0 + C$ and $C = 3$. Thus
$u^2 - 2u = t^2 + t + 3$; the quadratic formula gives
$u = 1 - \sqrt{t^2 + t + 4}$.

14. $\dfrac{dy}{dt} = \dfrac{ty + 3t}{t^2 + 1} = \dfrac{t(y+3)}{t^2+1}$,
$y(2) = 2$. $\displaystyle\int \frac{dy}{y+3} = \int \frac{t\,dt}{t^2+1} \;\Rightarrow$
$\ln|y+3| = \frac{1}{2}\ln(t^2+1) + C \;\Rightarrow\; y + 3 = A\sqrt{t^2+1}$.
$y(2) = 2 \;\Rightarrow\; 5 = A\sqrt{5} \;\Rightarrow\; A = \sqrt{5} \;\Rightarrow$
$y = -3 + \sqrt{5t^2 + 5}$.

15. $y' = y\sin x$, $y(0) = 1$. $\displaystyle\int \frac{dy}{y} = \int \sin x\, dx \quad\Leftrightarrow$
$\ln|y| = -\cos x + C \quad\Rightarrow\quad |y| = e^{-\cos x + C} \quad\Rightarrow$
$y(x) = Ae^{-\cos x}$. $y(0) = Ae^{-1} = 1 \quad\Leftrightarrow\quad A = e^1$, so
$y = e \cdot e^{-\cos x} = e^{1-\cos x}$.



16. Let $y = f(x)$. Then $\dfrac{dy}{dx} = x^3 y$ and $y(0) = 1$. $\dfrac{dy}{y} = x^3\, dx$
(if $y \neq 0$), so $\displaystyle\int \frac{dy}{y} = \int x^3\, dx$ and $\ln|y| = \frac{1}{4}x^4 + C$;
$y(0) = 1 \quad\Rightarrow\quad C = 0$, so $\ln|y| = \frac{1}{4}x^4$, $|y| = e^{x^4/4}$ and
$y = f(x) = e^{x^4/4}$ [since $y(0) = 1$].

17. Let $y = g(x)$. Then $\dfrac{dy}{dx} = y(1+y)$ and $y(0) = 1$.
$\displaystyle\int \frac{dy}{y(1+y)} = \int dx \quad\Rightarrow\quad \int\left(\frac{1}{y} - \frac{1}{1+y}\right) dy = \int dx$
$\Rightarrow\quad \ln|y| - \ln|1+y| = x + C \quad\Rightarrow\quad \left|\dfrac{y}{1+y}\right| = e^C e^x$
$\Rightarrow\quad \dfrac{y}{1+y} = Ae^x$. $y(0) = 1 \quad\Rightarrow\quad \frac{1}{2} = A$, so
$\dfrac{y}{1+y} = \dfrac{e^x}{2}$. Solve for y: $y = \dfrac{e^x}{2 - e^x}$.