



## Answers

**E** Click here for exercises.

1.  $-\cos x + \frac{1}{3} \cos^3 x + C$
2.  $\frac{1}{5} \sin^5 x - \frac{1}{7} \sin^7 x + C$
3.  $\frac{\pi}{4}$
4.  $\frac{\pi}{4}$
5.  $\frac{3}{2}x + \cos 2x - \frac{1}{8} \sin 4x + C$
6.  $-\frac{1}{2} \cos(x^2) + \frac{1}{6} \cos^3(x^2) + C$
7.  $\sqrt{x} + \frac{1}{2} \sin(2\sqrt{x}) + C$
8.  $\frac{2}{\sqrt{3}}$
9.  $\frac{64}{15}$
10.  $\frac{1}{128} \left( \sec^{-1} \frac{x}{4} + \frac{4\sqrt{x^2-16}}{x^2} \right) + C$
11.  $\frac{1}{\sqrt{3}} \ln \left| \frac{\sqrt{x^2+3}-\sqrt{3}}{x} \right| + C$
12.  $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1}$

**S** Click here for solutions.

13.  $1 + \frac{A}{x-1} + \frac{B}{x+1}$
14.  $\frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2}$
15.  $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1} + \frac{Ex+F}{x^2+x+1}$
16.  $\frac{1}{2}x^2 - x + \ln|x+1| + C$
17.  $x + \ln \frac{(x-1)^2}{|x|} + C$
18.  $\frac{1}{3} \ln \frac{5}{2}$
19.  $\frac{1}{3} \ln|x| - \ln|x+1| + \frac{2}{3} \ln|2x+3| + C$
20.  $\frac{1}{2} (1 - \ln 2)$
21.  $(x-1)^2 + \ln \sqrt{x^2+1} - 3 \tan^{-1} x + C$

## Solutions

**E** Click here for exercises.

- $$\begin{aligned} \int \sin^3 x \, dx &= \int (1 - \cos^2 x) \sin x \, dx \\ &= \int \sin x \, dx + \int \cos^2 x (-\sin x) \, dx \\ &= -\cos x + \frac{1}{3} \cos^3 x + C \end{aligned}$$

by the substitution  $u = \cos x$  in the second integral.
- Let  $u = \sin x \Rightarrow du = \cos x \, dx$ . Then

$$\begin{aligned} \int \sin^4 x \cos^3 x \, dx &= \int \sin^4 x (1 - \sin^2 x) \cos x \, dx \\ &= \int (u^4 - u^6) \, du = \frac{1}{5} u^5 - \frac{1}{7} u^7 + C \\ &= \frac{1}{5} \sin^5 x - \frac{1}{7} \sin^7 x + C \end{aligned}$$
- $$\begin{aligned} \int_0^{\pi/2} \sin^2 3x \, dx &= \int_0^{\pi/2} \frac{1}{2} (1 - \cos 6x) \, dx \\ &= \left[ \frac{1}{2} x - \frac{1}{12} \sin 6x \right]_0^{\pi/2} = \frac{\pi}{4} \end{aligned}$$
- $$\begin{aligned} \int_0^{\pi/2} \cos^2 x \, dx &= \int_0^{\pi/2} \frac{1}{2} (1 + \cos 2x) \, dx \\ &= \left[ \frac{1}{2} x + \frac{1}{4} \sin 2x \right]_0^{\pi/2} = \frac{\pi}{4} \end{aligned}$$
- $$\begin{aligned} \int (1 - \sin 2x)^2 \, dx &= \int (1 - 2 \sin 2x + \sin^2 2x) \, dx \\ &= \int \left[ 1 - 2 \sin 2x + \frac{1}{2} (1 - \cos 4x) \right] \, dx \\ &= \int \left[ \frac{3}{2} - 2 \sin 2x - \frac{1}{2} \cos 4x \right] \, dx \\ &= \frac{3}{2} x + \cos 2x - \frac{1}{8} \sin 4x + C \end{aligned}$$
- Let  $u = x^2 \Rightarrow du = 2x \, dx$ . Then

$$\begin{aligned} \int x \sin^3(x^2) \, dx &= \int \sin^3 u \cdot \frac{1}{2} \, du \\ &= \frac{1}{2} (-\cos u + \frac{1}{3} \cos^3 u) + C \\ &= -\frac{1}{2} \cos(x^2) + \frac{1}{6} \cos^3(x^2) + C \end{aligned}$$
- Let  $u = \sqrt{x}$  so that  $x = u^2$  and  $dx = 2u \, du$ . Then

$$\begin{aligned} \int \frac{\cos^2 \sqrt{x}}{\sqrt{x}} \, dx &= \int \frac{\cos^2 u}{u} 2u \, du = \int 2 \cos^2 u \, du \\ &= \int (1 + \cos 2u) \, du \\ &= u + \frac{1}{2} \sin 2u + C \\ &= \sqrt{x} + \frac{1}{2} \sin(2\sqrt{x}) + C \end{aligned}$$
- Let  $x = \sin \theta$ , where  $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ . Then  $dx = \cos \theta \, d\theta$  and  $\sqrt{1-x^2} = |\cos \theta| = \cos \theta$  (since  $\cos \theta > 0$  for  $\theta$  in  $[-\frac{\pi}{2}, \frac{\pi}{2}]$ ). Thus

$$\begin{aligned} \int_{1/2}^{\sqrt{3}/2} \frac{dx}{x^2 \sqrt{1-x^2}} &= \int_{\pi/6}^{\pi/3} \frac{\cos \theta \, d\theta}{\sin^2 \theta \cos \theta} \\ &= \int_{\pi/6}^{\pi/3} \csc^2 \theta \, d\theta = [-\cot \theta]_{\pi/6}^{\pi/3} \\ &= -\frac{1}{\sqrt{3}} - (-\sqrt{3}) \\ &= \frac{3}{\sqrt{3}} - \frac{1}{\sqrt{3}} = \frac{2}{\sqrt{3}} \end{aligned}$$

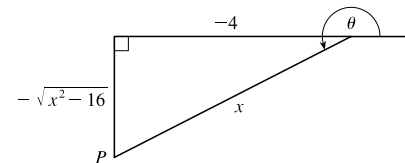
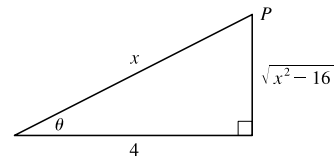
**A** Click here for answers.

- Let  $x = 2 \sin \theta$ ,  $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ . Then  $dx = 2 \cos \theta \, d\theta$  and  $\sqrt{4-x^2} = |2 \cos \theta| = 2 \cos \theta$ , so

$$\begin{aligned} \int_0^2 x^3 \sqrt{4-x^2} \, dx &= \int_0^{\pi/2} 8 \sin^3 \theta (2 \cos \theta) (2 \cos \theta) \, d\theta \\ &= 32 \int_0^{\pi/2} \cos^2 \theta (1 - \cos^2 \theta) \sin \theta \, d\theta \\ &= 32 \int_1^0 u^2 (1 - u^2) (-du) \quad (\text{where } u = \cos \theta) \\ &= 32 \int_0^1 (u^2 - u^4) \, du = 32 \left[ \frac{1}{3} u^3 - \frac{1}{5} u^5 \right]_0^1 \\ &= 32 \left( \frac{1}{3} - \frac{1}{5} \right) = \frac{64}{15} \end{aligned}$$
- Let  $x = 4 \sec \theta$ , where  $0 \leq \theta < \frac{\pi}{2}$  or  $\pi \leq \theta < \frac{3\pi}{2}$ . Then  $dx = 4 \sec \theta \tan \theta \, d\theta$  and  $\sqrt{x^2-16} = 4 |\tan \theta| = 4 \tan \theta$ . Thus

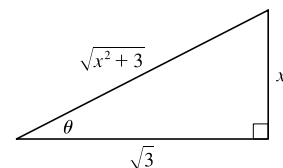
$$\begin{aligned} \int \frac{dx}{x^3 \sqrt{x^2-16}} &= \int \frac{4 \sec \theta \tan \theta \, d\theta}{64 \sec^3 \theta \cdot 4 \tan \theta} \\ &= \frac{1}{64} \int \cos^2 \theta \, d\theta = \frac{1}{128} \int (1 + \cos 2\theta) \, d\theta \\ &= \frac{1}{128} \left( \theta + \frac{1}{2} \sin 2\theta \right) + C = \frac{1}{128} (\theta + \sin \theta \cos \theta) + C \\ &= \frac{1}{128} \left( \sec^{-1} \frac{x}{4} + \frac{4\sqrt{x^2-16}}{x^2} \right) + C \end{aligned}$$

by the diagrams for  $0 \leq \theta < \frac{\pi}{2}$  and  $\pi \leq \theta < \frac{3\pi}{2}$ , where the labels of the legs in the second diagram indicate the  $x$ - and  $y$ -coordinates of  $P$  rather than the lengths of those sides. Henceforth we omit the second diagram from our solutions.



- Let  $x = \sqrt{3} \tan \theta$ , where  $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ . Then

$$\begin{aligned} \int \frac{dx}{x\sqrt{x^2+3}} &= \int \frac{\sqrt{3} \sec^2 \theta \, d\theta}{\sqrt{3} \tan \theta \sqrt{3} \sec \theta} \\ &= \frac{1}{\sqrt{3}} \int \csc \theta \, d\theta \\ &= \frac{1}{\sqrt{3}} \ln |\csc \theta - \cot \theta| + C \\ &= \frac{1}{\sqrt{3}} \ln \left| \frac{\sqrt{x^2+3} - \sqrt{3}}{x} \right| + C \end{aligned}$$



$$12. \frac{1}{x^4 - x^3} = \frac{1}{x^3(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1}$$

$$13. \frac{x^2 + 1}{x^2 - 1} = 1 + \frac{2}{(x-1)(x+1)} = 1 + \frac{A}{x-1} + \frac{B}{x+1}$$

$$14. \frac{x^3 - 4x^2 + 2}{(x^2 + 1)(x^2 + 2)} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{x^2 + 2}$$

$$15. \frac{1}{x^6 - x^3} = \frac{1}{x^3(x^3 - 1)} = \frac{1}{x^3(x-1)(x^2 + x + 1)}$$

$$= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1} + \frac{Ex + F}{x^2 + x + 1}$$

$$16. \int \frac{x^2}{x+1} dx = \int \left( x - 1 + \frac{1}{x+1} \right) dx$$

$$= \frac{1}{2}x^2 - x + \ln|x+1| + C$$

$$17. \frac{x^2 + 1}{x^2 - x} = 1 + \frac{x+1}{x(x-1)} = 1 - \frac{1}{x} + \frac{2}{x-1}, \text{ so}$$

$$\int \frac{x^2 + 1}{x^2 - x} dx = x - \ln|x| + 2 \ln|x-1| + C$$

$$= x + \ln \frac{(x-1)^2}{|x|} + C$$

$$18. \frac{1}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2} \Rightarrow$$

$1 = A(x-2) + B(x+1)$ . Taking  $x = -1$ , then  $x = 2$ , gives  $A = -\frac{1}{3}$ ,  $B = \frac{1}{3}$ . Hence

$$\int_3^7 \frac{dx}{(x+1)(x-2)} = \frac{1}{3} \int_3^7 \left[ \frac{1}{x-2} - \frac{1}{x+1} \right] dx$$

$$= \frac{1}{3} [\ln|x-2| - \ln|x+1|]_3^7$$

$$= \frac{1}{3} (\ln 5 - \ln 8 - \ln 1 + \ln 4)$$

$$= \frac{1}{3} \ln \frac{5}{2}$$

$$19. \frac{1}{x(x+1)(2x+3)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{2x+3} \Rightarrow$$

$1 = A(x+1)(2x+3) + B(x)(2x+3) + C(x)(x+1)$ .  
Set  $x = 0$  to get  $A = \frac{1}{3}$ . Take  $x = -1$  to get  $B = -1$ , and finally setting  $x = -\frac{3}{2}$  gives  $C = \frac{4}{3}$ . Now

$$\int \frac{dx}{x(x+1)(2x+3)}$$

$$= \int \left( \frac{1/3}{x} - \frac{1}{x+1} + \frac{4/3}{2x+3} \right) dx$$

$$= \frac{1}{3} \ln|x| - \ln|x+1| + \frac{2}{3} \ln|2x+3| + C$$

$$20. \frac{x^3}{x^2 + 1} = \frac{(x^3 + x) - x}{x^2 + 1} = x - \frac{x}{x^2 + 1}, \text{ so}$$

$$\int_0^1 \frac{x^3}{x^2 + 1} dx = \int_0^1 x dx - \int_0^1 \frac{x dx}{x^2 + 1}$$

$$= \left[ \frac{1}{2}x^2 \right]_0^1 - \frac{1}{2} \int_1^2 \frac{1}{u} du \text{ (where } u = x^2 + 1, du = 2x dx)$$

$$= \frac{1}{2} - \left[ \frac{1}{2} \ln u \right]_1^2 = \frac{1}{2} - \frac{1}{2} \ln 2 = \frac{1}{2} (1 - \ln 2)$$

$$21. \frac{3x^2 - 4x + 5}{(x-1)(x^2 + 1)} = \frac{A}{x-1} + \frac{Bx + C}{x^2 + 1} \Rightarrow$$

$3x^2 - 4x + 5 = A(x^2 + 1) + (Bx + C)(x-1)$ . Take  $x = 1$  to get  $4 = 2A$  or  $A = 2$ . Now

$$(Bx + C)(x-1) = 3x^2 - 4x + 5 - 2(x^2 + 1)$$

$$= x^2 - 4x + 3$$

Equating coefficients of  $x^2$  and then comparing the constant terms, we get  $B = 1$  and  $C = -3$ . Hence,

$$\int \frac{3x^2 - 4x + 5}{(x-1)(x^2 + 1)} dx = \int \left[ \frac{2}{x-1} + \frac{x-3}{x^2 + 1} \right] dx$$

$$= 2 \ln|x-1| + \int \frac{x dx}{x^2 + 1} - 3 \int \frac{dx}{x^2 + 1}$$

$$= 2 \ln|x-1| + \frac{1}{2} \ln(x^2 + 1) - 3 \tan^{-1} x + C$$

$$= \ln(x-1)^2 + \ln \sqrt{x^2 + 1} - 3 \tan^{-1} x + C$$