

## 5.1 Areas and Distances

**A** [Click here for answers.](#)

**S** [Click here for solutions.](#)

1. (a) Estimate the area under the graph of  $f(x) = x^3 + 2$  from  $x = -1$  to  $x = 2$  using three rectangles and right endpoints. Then improve your estimate by using six rectangles. Sketch the curve and the approximating rectangles.
- (b) Repeat part (a) using left endpoints.
- (c) Repeat part (a) using midpoints.
- (d) From your sketches in parts (a), (b), and (c), which appears to be the best estimate?

**2-4** Use Definition 2 to find an expression for the area under the graph of  $f$  as a limit. Do not evaluate the limit.

2.  $f(x) = \sqrt[3]{x}, \quad 0 \leq x \leq 8$

3.  $f(x) = 5 + \sqrt[3]{x}, \quad 1 \leq x \leq 8$

4.  $f(x) = x + \ln x, \quad 2 \leq x \leq 6$

5. Determine a region whose area is equal to

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{3}{n} \sqrt{1 + \frac{3i}{n}}$$

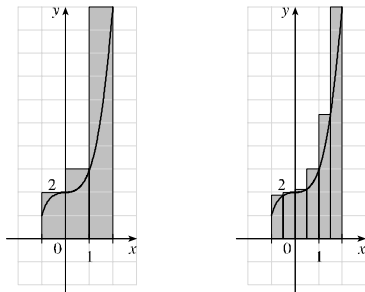
Do not evaluate the limit.

# Answers

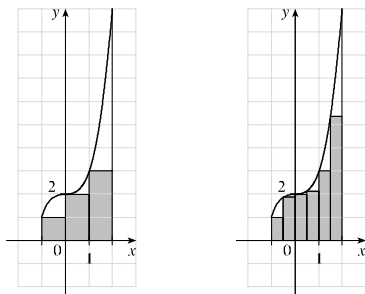
[E Click here for exercises.](#)

[S Click here for solutions.](#)

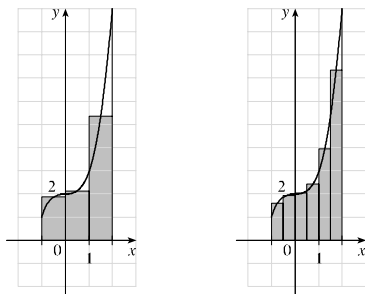
1. (a) 15, 12.1875



(b) 6, 7.6875



(c) 9.375, 9.65625



(d)  $M_6$

$$2. \lim_{n \rightarrow \infty} \frac{8}{n} \sum_{i=1}^n \sqrt[3]{\frac{8i}{n}}$$

$$3. \lim_{n \rightarrow \infty} \frac{7}{n} \sum_{i=1}^n \left( 5 + \sqrt[3]{1 + \frac{7i}{n}} \right)$$

$$4. \lim_{n \rightarrow \infty} \frac{4}{n} \sum_{i=1}^n \left[ 2 + \frac{4i}{n} + \ln \left( 2 + \frac{4i}{n} \right) \right]$$

5. The area of the region under the graph of  $\sqrt{x}$  on the interval  $[1, 4]$ , or the area of the region lying under the graph of  $\sqrt{x+1}$  on the interval  $[0, 3]$

## Solutions

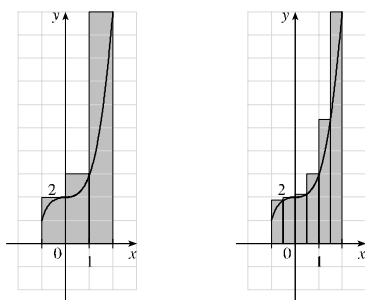
[Click here for exercises.](#)

$$1. (a) f(x) = x^3 + 2 \text{ and } \Delta x = \frac{2 - (-1)}{3} = 1 \Rightarrow$$

$$R_3 = 1 \cdot f(0) + 1 \cdot f(1) + 1 \cdot f(2) \\ = 1 \cdot 2 + 1 \cdot 3 + 1 \cdot 10 = 15$$

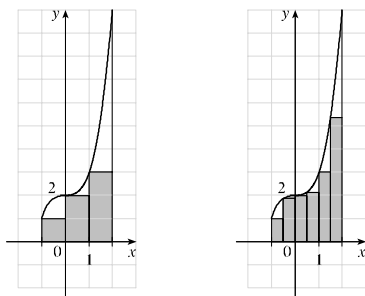
$$\Delta x = \frac{2 - (-1)}{6} = 0.5 \Rightarrow$$

$$R_6 = 0.5 [f(-0.5) + f(0) + f(0.5) \\ + f(1) + f(1.5) + f(2)] \\ = 0.5 (1.875 + 2 + 2.125 + 3 + 5.375 + 10) \\ = 0.5 (24.375) = 12.1875$$



$$(b) L_3 = 1 \cdot f(-1) + 1 \cdot f(0) + 1 \cdot f(1) \\ = 1 \cdot 1 + 1 \cdot 2 + 1 \cdot 3 = 6$$

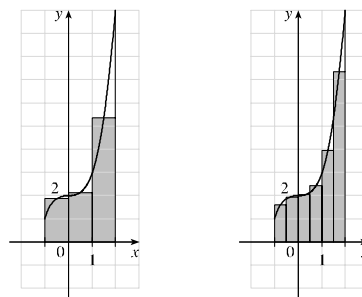
$$L_6 = 0.5 [f(-1) + f(-0.5) + f(0) \\ + f(0.5) + f(1) + f(1.5)] \\ = 0.5 (1 + 1.875 + 2 + 2.125 + 3 + 5.375) \\ = 0.5 (15.375) = 7.6875$$



$$(c) M_3 = 1 \cdot f(-0.5) + 1 \cdot f(0.5) + 1 \cdot f(1.5) \\ = 1 \cdot 1.875 + 1 \cdot 2.125 + 1 \cdot 5.375 = 9.375$$

$$M_6 = 0.5 [f(-0.75) + f(-0.25) + f(0.25) \\ + f(0.75) + f(1.25) + f(1.75)] \\ = 0.5 (1.578125 + 1.984375 + 2.015625 \\ + 2.421875 + 3.953125 + 7.359375) \\ = 0.5 (19.3125) = 9.65625$$

[Click here for answers.](#)



(d)  $M_6$  appears to be the best estimate.

$$2. f(x) = \sqrt[3]{x}, 0 \leq x \leq 8 \Rightarrow \Delta x = \frac{8-0}{n} = \frac{8}{n}, \\ x_i = 0 + i \Delta x = \frac{8i}{n} \Rightarrow$$

$$R_n = f(x_1) \Delta x + f(x_2) \Delta x + \cdots + f(x_n) \Delta x \\ = \sum_{i=1}^n f(x_i) \Delta x = \sum_{i=1}^n \sqrt[3]{\frac{8i}{n}} \cdot \frac{8}{n}$$

$$\text{Thus, } A = \lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} \frac{8}{n} \sum_{i=1}^n \sqrt[3]{\frac{8i}{n}}$$

$$3. f(x) = 5 + \sqrt[3]{x}, 1 \leq x \leq 8 \Rightarrow \Delta x = \frac{8-1}{n} = \frac{7}{n}, \\ x_i = 1 + i \Delta x = 1 + \frac{7i}{n} \Rightarrow$$

$$R_n = f(x_1) \Delta x + f(x_2) \Delta x + \cdots + f(x_n) \Delta x \\ = \sum_{i=1}^n f(x_i) \Delta x = \sum_{i=1}^n \left( 5 + \sqrt[3]{1 + \frac{7i}{n}} \right) \cdot \frac{7}{n}$$

$$\text{Thus, } A = \lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} \frac{7}{n} \sum_{i=1}^n \left( 5 + \sqrt[3]{1 + \frac{7i}{n}} \right)$$

$$4. f(x) = x + \ln x, 2 \leq x \leq 6 \Rightarrow \Delta x = \frac{6-2}{n} = \frac{4}{n}, \\ x_i = 2 + i \Delta x = 2 + \frac{4i}{n} \Rightarrow$$

$$R_n = f(x_1) \Delta x + f(x_2) \Delta x + \cdots + f(x_n) \Delta x \\ = \sum_{i=1}^n f(x_i) \Delta x = \sum_{i=1}^n \left[ 2 + \frac{4i}{n} + \ln \left( 2 + \frac{4i}{n} \right) \right] \cdot \frac{4}{n}$$

Thus,

$$A = \lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} \frac{4}{n} \sum_{i=1}^n \left[ 2 + \frac{4i}{n} + \ln \left( 2 + \frac{4i}{n} \right) \right]$$

5. The two most obvious ways to interpret

$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{3}{n} \sqrt{1 + \frac{3i}{n}}$  are: as the area of the region under the

graph of  $\sqrt{x}$  on the interval  $[1, 4]$ , or as the area of the region

lying under the graph of  $\sqrt{x+1}$  on the interval  $[0, 3]$ , since

for  $y = \sqrt{x+1}$  on  $[0, 3]$  with partition points  $x_i = \frac{i}{3n}$ ,

$\Delta x = \frac{1}{3n}$  and  $x_i^* = x_i$ , the expression for the area is

$$A = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x = \lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt{1 + \frac{3i}{n}} \left( \frac{3}{n} \right).$$