4.5 Indeterminate Forms and L'Hospital's Rule

A Click here for answers.

S Click here for solutions.

1–45 Find the limit. Use l'Hospital's Rule where appropriate. If there is a more elementary method, use it. If l'Hospital's Rule doesn't apply, explain why.

1. $\lim_{x \to 2} \frac{x-2}{x^2-4}$

2. $\lim_{x \to 1} \frac{x^2+3x-4}{x-1}$

3. $\lim_{x \to -1} \frac{x^6-1}{x^4-1}$

4. $\lim_{x \to 0} \frac{\tan x}{x+\sin x}$

5. $\lim_{x \to 0} \frac{e^x-1}{\sin x}$

6. $\lim_{x \to 0} \frac{x+\tan x}{\sin x}$

7. $\lim_{x \to 0} \frac{\sin x}{x^3}$

8. $\lim_{x \to \pi} \frac{\tan x}{x}$

9. $\lim_{x \to 3\pi/2} \frac{\cos x}{x-(3\pi/2)}$

10. $\lim_{t \to 16} \frac{\sqrt[4]{t}-2}{t-16}$

11. $\lim_{x \to a} \frac{\sqrt[3]{x}-\sqrt[3]{a}}{x-a}$

12. $\lim_{x \to 0} \frac{6^x-2^x}{x}$

13. $\lim_{x \to \infty} \frac{(\ln x)^3}{x^2}$

14. $\lim_{x \to 0} \frac{\sin x}{e^x}$

15. $\lim_{x \to 0} \frac{\tan \alpha x}{x}$

16. $\lim_{x \to 0} \frac{\sin^2 x}{\tan(x^2)}$

17. $\lim_{x \to \infty} \frac{\ln \ln x}{\sqrt{x}}$

18. $\lim_{x \to \infty} \frac{\ln(1+e^x)}{5x}$

19. $\lim_{x \to 0} \frac{\tan^{-1}(2x)}{3x}$

20. $\lim_{x \to 0} \frac{x}{\sin^{-1}(3x)}$

21. $\lim_{x \to 0} \frac{\sin mx}{\sin nx}$

22. $\lim_{x \to 0} \frac{\sin^{10} x}{\sin(x^{10})}$

23. $\lim_{x \to 0} \frac{x+\sin 3x}{x-\sin 3x}$

24. $\lim_{x \to 0} \frac{e^{4x}-1}{\cos x}$

25. $\lim_{x \to 0} \frac{\tan x-\sin x}{x^3}$

26. $\lim_{x \to 0} \frac{x+\tan 2x}{x-\tan 2x}$

27. $\lim_{x \to 0} \frac{\tan 2x}{\tanh 3x}$

28. $\lim_{x \to 0} \frac{2x-\sin^{-1}x}{2x+\cos^{-1}x}$

29. $\lim_{x \to 0} \frac{2x-\sin^{-1}x}{2x+\tan^{-1}x}$

30. $\lim_{x \to -\infty} xe^x$

31. $\lim_{x \to \infty} e^{-x} \ln x$

32. $\lim_{x \to (\pi/2)^-} \sec 7x \cos 3x$

33. $\lim_{x \to 0^+} \sqrt{x} \sec x$

34. $\lim_{x \to \pi} (x-\pi) \cot x$

35. $\lim_{x \to 1^+} (x-1)\tan(\pi x/2)$

36. $\lim_{x \to 0} \left(\frac{1}{x^4}-\frac{1}{x^2}\right)$

37. $\lim_{x \to \infty} \left(x-\sqrt{x^2-1}\right)$

38. $\lim_{x \to \infty} \left(\sqrt{x^2+x+1}-\sqrt{x^2-x}\right)$

39. $\lim_{x \to \infty} \left(\frac{1}{x}-\frac{1}{e^x-1}\right)$

40. $\lim_{x \to \infty} \left(\frac{x^3}{x^2-1}-\frac{x^3}{x^2+1}\right)$

41. $\lim_{x \to 0^+} (\sin x)^{\tan x}$

42. $\lim_{x \to \infty} \left(1+\frac{1}{x^2}\right)^x$

43. $\lim_{x \to 0^+} (\cot x)^{\sin x}$

44. $\lim_{x \to \infty} \left(1+\frac{1}{x}\right)^{x^2}$

45. $\lim_{x \to 0^-} (-\ln x)^x$

Answers

E Click here for exercises.

S Click here for solutions.

1. $\frac{1}{4}$ **2.** 5

3. $\frac{3}{2}$ **4.** $\frac{1}{2}$

5. 1 **6.** 2

7. ∞ **8.** 0

9. 1 **10.** $\frac{1}{32}$

11. $\frac{1}{3a^{2/3}}$ **12.** $\ln 3$

13. 0 **14.** 0

15. α **16.** 1

17. 0 **18.** $\frac{1}{5}$

19. $\frac{2}{3}$ **20.** $\frac{1}{3}$

21. $\frac{m}{n}$ **22.** 1

23. -2 **24.** 0

25. $\frac{1}{2}$ **26.** -3

27. $\frac{2}{3}$ **28.** 0

29. $\frac{1}{3}$ **30.** 0

31. 0 **32.** $\frac{3}{7}$

33. 0 **34.** 1

35. $-\frac{2}{\pi}$ **36.** ∞

37. 0 **38.** 1

39. 0 **40.** 0

41. 1 **42.** 1

43. 1 **44.** ∞

45. 1

Solutions

E Click here for exercises.

A Click here for answers.

The use of l'Hospital's Rule is indicated by an H above the equal sign: $\overset{H}{=}$.

1. $\lim_{x\to 2}\frac{x-2}{x^2-4}=\lim_{x\to 2}\frac{x-2}{(x-2)(x+2)}=\lim_{x\to 2}\frac{1}{x+2}=\frac{1}{4}$

2. $\lim_{x\to 1}\frac{x^2+3x-4}{x-1}=\lim_{x\to 1}\frac{(x-1)(x+4)}{x-1}$
$=\lim_{x\to 1}(x+4)=5$

3. $\lim_{x\to -1}\frac{x^6-1}{x^4-1}\overset{H}{=}\lim_{x\to -1}\frac{6x^5}{4x^3}=\frac{-6}{-4}=\frac{3}{2}$

4. $\lim_{x\to 0}\frac{\tan x}{x+\sin x}\overset{H}{=}\lim_{x\to 0}\frac{\sec^2 x}{1+\cos x}=\frac{1}{1+1}=\frac{1}{2}$

5. $\lim_{x\to 0}\frac{e^x-1}{\sin x}\overset{H}{=}\lim_{x\to 0}\frac{e^x}{\cos x}=\frac{1}{1}=1$

6. $\lim_{x\to 0}\frac{x+\tan x}{\sin x}\overset{H}{=}\lim_{x\to 0}\frac{1+\sec^2 x}{\cos x}=\frac{1+1^2}{1}=2$

7. $\lim_{x\to 0}\frac{\sin x}{x^3}\overset{H}{=}\lim_{x\to 0}\frac{\cos x}{3x^2}=\infty$

8. $\lim_{x\to \pi}\frac{\tan x}{x}=\frac{\tan\pi}{\pi}=\frac{0}{\pi}=0$. L'Hospital's Rule does not apply because the denominator doesn't approach 0.

9. $\lim_{x\to 3\pi/2}\frac{\cos x}{x-3\pi/2}\overset{H}{=}\lim_{x\to 3\pi/2}\frac{-\sin x}{1}=-\sin\frac{3\pi}{2}=1$

10. $\lim_{t\to 16}\frac{\sqrt[4]{t}-2}{t-16}=\lim_{t\to 16}\frac{\sqrt[4]{t}-2}{(\sqrt{t}+4)(\sqrt{t}-4)}$
$=\lim_{t\to 16}\frac{\sqrt[4]{t}-2}{(\sqrt{t}+4)(\sqrt[4]{t}+2)(\sqrt[4]{t}-2)}$
$=\lim_{t\to 16}\frac{1}{(\sqrt{t}+4)(\sqrt[4]{t}+2)}$
$=\frac{1}{(4+4)(2+2)}=\frac{1}{32}$

11. $\lim_{x\to a}\frac{x^{1/3}-a^{1/3}}{x-a}\overset{H}{=}\lim_{x\to a}\frac{(1/3)x^{-2/3}}{1}=\frac{1}{3a^{2/3}}$

12. $\lim_{x\to 0}\frac{6^x-2^x}{x}\overset{H}{=}\lim_{x\to 0}\frac{6^x(\ln 6)-2^x(\ln 2)}{1}$
$=\ln 6-\ln 2=\ln\frac{6}{2}=\ln 3$

13. $\lim_{x\to\infty}\frac{(\ln x)^3}{x^2}\overset{H}{=}\lim_{x\to\infty}\frac{3(\ln x)^2(1/x)}{2x}=\lim_{x\to\infty}\frac{3(\ln x)^2}{2x^2}$
$\overset{H}{=}\lim_{x\to\infty}\frac{6(\ln x)(1/x)}{4x}=\lim_{x\to\infty}\frac{3\ln x}{2x^2}$
$\overset{H}{=}\lim_{x\to\infty}\frac{3/x}{4x}=\lim_{x\to\infty}\frac{3}{4x^2}=0$

14. $\lim_{x\to 0}\frac{\sin x}{e^x}=\frac{0}{1}=0$. L'Hospital's Rule does not apply.

15. $\lim_{x\to 0}\frac{\tan\alpha x}{x}\overset{H}{=}\lim_{x\to 0}\frac{\alpha\sec^2\alpha x}{1}=\alpha$

16. $\lim_{x\to 0}\frac{\sin^2 x}{\tan(x^2)}\overset{H}{=}\lim_{x\to 0}\frac{2\sin x\cos x}{2x\sec^2(x^2)}$
$=\lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}\frac{\cos x}{\sec^2(x^2)}=1\cdot 1=1$

17. $\lim_{x\to\infty}\frac{\ln\ln x}{\sqrt{x}}\overset{H}{=}\lim_{x\to\infty}\frac{1/(x\ln x)}{1/(2\sqrt{x})}=\lim_{x\to\infty}\frac{2}{\sqrt{x}\ln x}=0$

18. $\lim_{x\to\infty}\frac{\ln(1+e^x)}{5x}\overset{H}{=}\lim_{x\to\infty}\frac{e^x/(1+e^x)}{5}$
$=\lim_{x\to\infty}\frac{e^x}{5(1+e^x)}\overset{H}{=}\lim_{x\to\infty}\frac{e^x}{5e^x}=\frac{1}{5}$

19. $\lim_{x\to 0}\frac{\tan^{-1}(2x)}{3x}\overset{H}{=}\lim_{x\to 0}\frac{2/(1+4x^2)}{3}=\frac{2}{3}$

20. $\lim_{x\to 0}\frac{x}{\sin^{-1}(3x)}\overset{H}{=}\lim_{x\to 0}\frac{1}{3/\sqrt{1-(3x)^2}}$
$=\lim_{x\to 0}\frac{1}{3}\sqrt{1-9x^2}=\frac{1}{3}$

21. $\lim_{x\to 0}\frac{\sin mx}{\sin nx}\overset{H}{=}\lim_{x\to 0}\frac{m\cos mx}{n\cos nx}=\frac{m}{n}$

22. $\lim_{x\to 0}\frac{\sin^{10}x}{\sin(x^{10})}\overset{H}{=}\lim_{x\to 0}\frac{10\sin^9 x\cos x}{10x^9\cos(x^{10})}$
$=\left[\lim_{x\to 0}\frac{\sin x}{x}\right]^9\lim_{x\to 0}\frac{\cos x}{\cos(x^{10})}$
$=1^9\cdot 1=1$

23. $\lim_{x\to 0}\frac{x+\sin 3x}{x-\sin 3x}\overset{H}{=}\lim_{x\to 0}\frac{1+3\cos 3x}{1-3\cos 3x}=\frac{1+3}{1-3}=-2$

24. $\lim_{x\to 0}\frac{e^{4x}-1}{\cos x}=\frac{0}{1}=0$

25. $\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}\overset{H}{=}\lim_{x\to 0}\frac{\sec^2 x-\cos x}{3x^2}$
$\overset{H}{=}\lim_{x\to 0}\frac{2\sec^2 x\tan x+\sin x}{6x}$
$\overset{H}{=}\lim_{x\to 0}\frac{4\sec^2 x\tan^2 x+2\sec^4 x+\cos x}{6}$
$=\frac{0+2+1}{6}=\frac{1}{2}$

26. $\lim_{x\to 0}\frac{x+\tan 2x}{x-\tan 2x}\overset{H}{=}\lim_{x\to 0}\frac{1+2\sec^2 2x}{1-2\sec^2 2x}$
$=\frac{1+2(1)^2}{1-2(1)^2}=-3$

27. $\lim_{x\to 0}\frac{\tan 2x}{\tanh 3x}\overset{H}{=}\lim_{x\to 0}\frac{2\sec^2 2x}{3\,\mathrm{sech}^2\,3x}=\frac{2}{3}$

28. $\lim_{x\to 0}\frac{2x-\sin^{-1}x}{2x+\cos^{-1}x}=\frac{2(0)-0}{2(0)+\pi/2}=0$. L'Hospital's Rule does not apply.

29. $\lim\limits_{x\to 0}\dfrac{2x-\sin^{-1}x}{2x+\tan^{-1}x}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{2-1/\sqrt{1-x^2}}{2+1/(1+x^2)}=\dfrac{2-1}{2+1}=\dfrac{1}{3}$

30. $\lim\limits_{x\to -\infty}xe^x=\lim\limits_{x\to -\infty}\dfrac{x}{e^{-x}}\overset{\text{H}}{=}\lim\limits_{x\to -\infty}\dfrac{1}{-e^{-x}}=\lim\limits_{x\to -\infty}-e^x=0$

31. $\lim\limits_{x\to\infty}e^{-x}\ln x=\lim\limits_{x\to\infty}\dfrac{\ln x}{e^x}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{1/x}{e^x}=\lim\limits_{x\to\infty}\dfrac{1}{xe^x}=0$

32. $\lim\limits_{x\to(\pi/2)^-}\sec 7x\cos 3x=\lim\limits_{x\to(\pi/2)^-}\dfrac{\cos 3x}{\cos 7x}\overset{\text{H}}{=}\lim\limits_{x\to(\pi/2)^-}\dfrac{-3\sin 3x}{-7\sin 7x}=\dfrac{3(-1)}{7(-1)}=\dfrac{3}{7}$

33. $\lim\limits_{x\to 0^+}\sqrt{x}\sec x=0\cdot 1=0$

34. $\lim\limits_{x\to\pi}(x-\pi)\cot x=\lim\limits_{x\to\pi}\dfrac{x-\pi}{\tan x}\overset{\text{H}}{=}\lim\limits_{x\to\pi}\dfrac{1}{\sec^2 x}=\dfrac{1}{(-1)^2}=1$

35. $\lim\limits_{x\to 1^+}(x-1)\tan(\pi x/2)=\lim\limits_{x\to 1^+}\dfrac{x-1}{\cot(\pi x/2)}\overset{\text{H}}{=}\lim\limits_{x\to 1^+}\dfrac{1}{-\csc^2(\pi x/2)\frac{\pi}{2}}=-\dfrac{2}{\pi}$

36. $\lim\limits_{x\to 0}\left(\dfrac{1}{x^4}-\dfrac{1}{x^2}\right)=\lim\limits_{x\to 0}\dfrac{1-x^2}{x^4}=\infty$

37. $\lim\limits_{x\to\infty}\left(x-\sqrt{x^2-1}\right)$

$$=\lim_{x\to\infty}\left(x-\sqrt{x^2-1}\right)\frac{x+\sqrt{x^2-1}}{x+\sqrt{x^2-1}}=\lim_{x\to\infty}\frac{x^2-(x^2-1)}{x+\sqrt{x^2-1}}=\lim_{x\to\infty}\frac{1}{x+\sqrt{x^2-1}}=0$$

38. $\lim\limits_{x\to\infty}\left(\sqrt{x^2+x+1}-\sqrt{x^2-x}\right)$

$$=\lim_{x\to\infty}\left(\sqrt{x^2+x+1}-\sqrt{x^2-x}\right)\frac{\sqrt{x^2+x+1}+\sqrt{x^2-x}}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}$$

$$=\lim_{x\to\infty}\frac{(x^2+x+1)-(x^2-x)}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}=\lim_{x\to\infty}\frac{2x+1}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}$$

$$=\lim_{x\to\infty}\frac{2+1/x}{\sqrt{1+1/x+1/x^2}+\sqrt{1-1/x}}=\frac{2}{1+1}=1$$

39. $\lim\limits_{x\to\infty}\left(\dfrac{1}{x}-\dfrac{1}{e^x-1}\right)=\lim\limits_{x\to\infty}\dfrac{1}{x}-\lim\limits_{x\to\infty}\dfrac{1}{e^x-1}$ (since both limits exist) $=0-0=0$

40. $\lim\limits_{x\to\infty}\left(\dfrac{x^3}{x^2-1}-\dfrac{x^3}{x^2+1}\right)$

$$=\lim_{x\to\infty}\frac{x^3(x^2+1)-x^3(x^2-1)}{(x^2-1)(x^2+1)}=\lim_{x\to\infty}\frac{2x^3}{x^4-1}=\lim_{x\to\infty}\frac{2/x}{1-1/x^4}=0$$

41. $y=(\sin x)^{\tan x}\Rightarrow\ln y=\tan x\ln(\sin x)$, so

$$\lim_{x\to 0^+}\ln y=\lim_{x\to 0^+}\tan x\ln(\sin x)=\lim_{x\to 0^+}\frac{\ln(\sin x)}{\cot x}\overset{\text{H}}{=}\lim_{x\to 0^+}\frac{(\cos x)/\sin x}{-\csc^2 x}=\lim_{x\to 0^+}(-\sin x\cos x)=0\Rightarrow$$

$\lim\limits_{x\to 0^+}(\sin x)^{\tan x}=\lim\limits_{x\to 0^+}e^{\ln y}=e^0=1.$

42. Let $y=\left(1+\dfrac{1}{x^2}\right)^x$. Then $\ln y=x\ln\left(1+\dfrac{1}{x^2}\right)\Rightarrow$

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}x\ln\left(1+\frac{1}{x^2}\right)=\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x^2}\right)}{1/x}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{\left(-\frac{2}{x^3}\right)\Big/\left(1+\frac{1}{x^2}\right)}{-1/x^2}=\lim_{x\to\infty}\frac{2/x}{1+1/x^2}=0,$$

so $\lim\limits_{x\to\infty}(1+1/x^2)^x=\lim\limits_{x\to\infty}e^{\ln y}=e^0=1.$

43. $y=(\cot x)^{\sin x}\Rightarrow\ln y=\sin x\ln(\cot x)\Rightarrow$

$$\lim_{x\to 0^+}\ln y=\lim_{x\to 0^+}\frac{\ln(\cot x)}{\csc x}\overset{\text{H}}{=}\lim_{x\to 0^+}\frac{(-\csc^2 x)/\cot x}{-\csc x\cot x}=\lim_{x\to 0^+}\frac{\csc x}{\cot^2 x}=\lim_{x\to 0^+}\frac{\sin x}{\cos^2 x}=0$$

so $\lim\limits_{x\to 0^+}(\cot x)^{\sin x}=\lim\limits_{x\to 0^+}e^{\ln y}=e^0=1.$

44. Let $y=(1+1/x)^{x^2}$. Then $\ln y=x^2\ln(1+1/x)\Rightarrow$

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}x^2\ln(1+1/x)=\lim_{x\to\infty}\frac{\ln(1+1/x)}{1/x^2}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{(-1/x^2)/(1+1/x)}{-2/x^3}=\lim_{x\to\infty}\frac{x}{2(1+1/x)}=\infty\Rightarrow$$

$\lim\limits_{x\to\infty}(1+1/x)^{x^2}=\lim\limits_{x\to\infty}e^{\ln y}=\infty.$

45. $y=(-\ln x)^x\Rightarrow\ln y=x\ln(-\ln x)$, so

$$\lim_{x\to 0^+}\ln y=\lim_{x\to 0^+}x\ln(-\ln x)=\lim_{x\to 0^+}\frac{\ln(-\ln x)}{1/x}\overset{\text{H}}{=}\lim_{x\to 0^+}\frac{(1/-\ln x)(-1/x)}{-1/x^2}=\lim_{x\to 0^+}\frac{-x}{\ln x}=0\Rightarrow$$

$\lim\limits_{x\to 0^+}(-\ln x)^x=e^0=1.$