

4.2 Maximum and Minimum Values

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1–4 ||| Sketch the graph of a function f that is continuous on $[0, 3]$ and has the given properties.

- Absolute maximum at 0, absolute minimum at 3, local minimum at 1, local maximum at 2
- Absolute maximum at 1, absolute minimum at 2
- 2 is a critical number, but f has no local maximum or minimum
- Absolute minimum at 0, absolute maximum at 2, local maxima at 1 and 2, local minimum at 1.5

5–18 ||| Sketch the graph of f by hand and use your sketch to find the absolute and local maximum and minimum values of f . (Use the graphs and transformations of Sections 1.2 and 1.3).

- $f(x) = 1 + 2x; \quad x \geq -1$
- $f(x) = 4x - 1; \quad x \leq 8$
- $f(x) = 1 - x^2; \quad 0 < x < 1$
- $f(x) = 1 - x^2; \quad 0 < x \leq 1$
- $f(x) = 1 - x^2; \quad 0 \leq x < 1$
- $f(x) = 1 - x^2; \quad 0 \leq x \leq 1$
- $f(x) = 1 - x^2; \quad -2 \leq x \leq 1$
- $f(x) = |x|; \quad -2 \leq x \leq 1$
- $f(x) = |4x - 1|; \quad 0 \leq x \leq 2$
- $f(\theta) = \cos(\theta/2); \quad -\pi < \theta < \pi$
- $f(\theta) = \sec \theta; \quad -\pi/2 < \theta \leq \pi/3$
- $f(x) = x^5$
- $f(x) = 2 - x^4$
- $f(x) = 1 - e^{-x}, \quad x \geq 0$

19–34 ||| Find the critical numbers of the function.

- $f(x) = 2x - 3x^2$
- $f(x) = x^3 - 3x + 1$
- $g(x) = \sqrt[3]{x}$
- $f(x) = 5 + 6x - 2x^3$
- $f(t) = 2t^3 + 3t^2 + 6t + 4$
- $f(x) = 5 + 8x$
- $f(t) = t^3 + 6t^2 + 3t - 1$
- $g(x) = |x + 1|$

S [Click here for solutions.](#)

- $f(x) = 4x^3 - 9x^2 - 12x + 3$
- $s(t) = 2t^3 + 3t^2 - 6t + 4$
- $s(t) = t^4 + 4t^3 + 2t^2$
- $f(r) = \frac{r}{r^2 + 1}$
- $g(\theta) = \theta + \sin \theta$
- $T(x) = x^2(2x - 1)^{2/3}$
- $f(\theta) = \sin^2(2\theta)$
- $V(x) = x\sqrt{x - 2}$

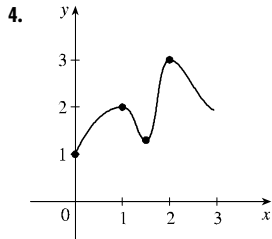
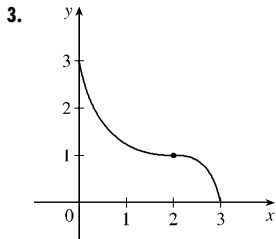
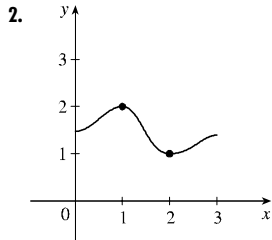
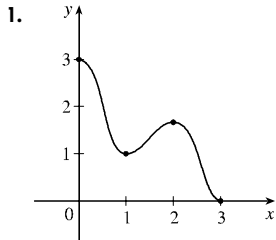
35–45 ||| Find the absolute maximum and absolute minimum values of f on the given interval.

- $f(x) = x^2 - 2x + 2, \quad [0, 3]$
- $f(x) = 1 - 2x - x^2, \quad [-4, 1]$
- $f(x) = x^3 - 12x + 1, \quad [-3, 5]$
- $f(x) = 4x^3 - 15x^2 + 12x + 7, \quad [0, 3]$
- $f(x) = 2x^3 + 3x^2 + 4, \quad [-2, 1]$
- $f(x) = 18x + 15x^2 - 4x^3, \quad [-3, 4]$
- $f(x) = x^4 - 4x^2 + 2, \quad [-3, 2]$
- $f(x) = 3x^5 - 5x^3 - 1, \quad [-2, 2]$
- $f(x) = x^2 + 2/x, \quad [\frac{1}{2}, 2]$
- $f(x) = \sqrt{9 - x^2}, \quad [-1, 2]$
- $f(x) = \frac{x}{x + 1}, \quad [1, 2]$

- Use a graph to estimate the critical numbers of $f(x) = x^4 - 3x^2 + x$ correct to one decimal place.
- (a) Use a graph to estimate the absolute maximum and minimum values of $f(x) = x^4 - 3x^3 + 3x^2 - x$ on the interval $[0, 2]$, correct to two decimal places.
(b) Use calculus to find the exact maximum and minimum values.
- Show that 0 is a critical number of the function $f(x) = x^5$ but f has neither a local maximum nor a local minimum at 0.
- Sketch the graph of a function on $[0, 1]$ that is discontinuous and has both an absolute maximum and an absolute minimum.

Answers

E Click here for exercises.

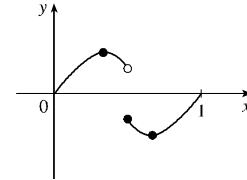


5. Abs. min. $f(-1) = -1$ 6. Abs. max. $f(8) = 31$
 7. None 8. Abs. min. $f(1) = 0$ 9. Abs. max. $f(0) = 1$
 10. Abs. max. $f(0) = 1$; abs. min. $f(1) = 0$
 11. Abs. and loc. max. $f(0) = 1$; abs. min. $f(-2) = -3$
 12. Abs. max. $f(-2) = 2$; abs. and loc. min $f(0) = 0$
 13. Abs. max. $f(2) = 7$; abs. and loc. min. $f(\frac{1}{4}) = 0$
 14. Abs. and loc. max. $f(0) = 1$
 15. Abs. and loc. min. $f(0) = 1$ 16. None
 17. Abs. and loc. max. $f(0) = 2$ 18. Abs. min. $f(0) = 0$
 19. $\frac{1}{3}$ 20. None 21. ± 1 22. $-2 \pm \sqrt{3}$
 23. 0 24. -1 25. ± 1 26. None
 27. $-\frac{1}{2}, 2$ 28. $(-1 \pm \sqrt{5})/2$ 29. $(-3 \pm \sqrt{5})/2$
 30. ± 1 31. $n\pi/4$, n an integer
 32. $(2n + 1)\pi$, n an integer 33. 2 34. $0, \frac{1}{2}, \frac{3}{8}$
 35. Abs. max. $f(3) = 5$; abs. min. $f(1) = 1$
 36. Abs. max. $f(-1) = 2$; abs. min. $f(-4) = -7$
 37. Abs. max. $f(5) = 66$; abs. min. $f(2) = -15$
 38. Abs. max. $f(3) = 16$; abs. min. $f(2) = 3$
 39. Abs. max. $f(1) = 9$; abs. min. $f(-2) = 0$
 40. Abs. max. $f(-3) = 189$; abs. min. $f(-\frac{1}{2}) = -\frac{19}{4}$
 41. Abs. max. $f(-3) = 47$; abs. min. $f(\pm\sqrt{2}) = -2$
 42. Abs. max. $f(2) = 55$; abs. min. $f(-2) = -57$
 43. Abs. max. $f(2) = 5$; abs. min. $f(1) = 3$

S Click here for solutions.

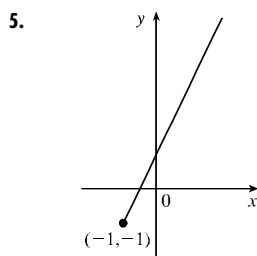
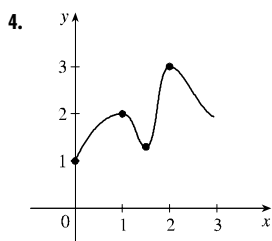
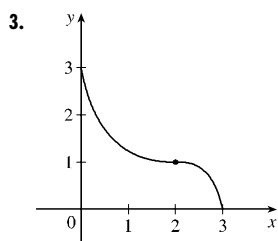
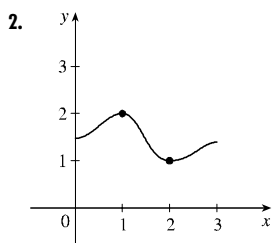
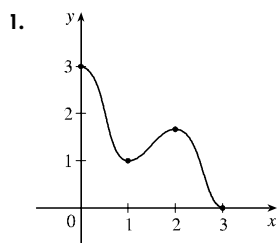
44. Abs. max. $f(0) = 3$; abs. min. $f(2) = \sqrt{5}$
 45. Abs. max. $f(2) = \frac{2}{3}$; abs. min. $f(1) = \frac{1}{2}$
 46. -1.3, 0.2, 1.1
 47. (a) 2, -0.11 (b) 2, $-\frac{27}{256}$

49.

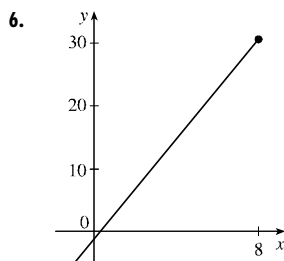



Solutions

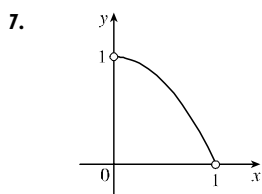
E [Click here for exercises.](#)



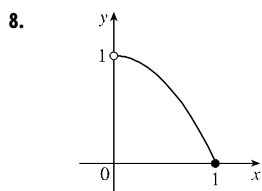
$f(x) = 1 + 2x, x \geq -1$.
Absolute minimum
 $f(-1) = -1$; no local
minimum. No local or absolute
maximum.



$f(x) = 4x - 1, x \leq 8$.
Absolute maximum $f(8) = 31$;
no local maximum. No local or
absolute minimum.

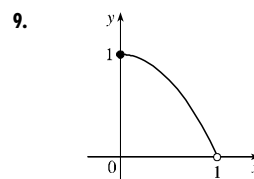


$f(x) = 1 - x^2, 0 < x < 1$. No
maximum or minimum.

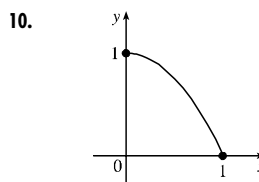


$f(x) = 1 - x^2, 0 < x \leq 1$.
Absolute minimum $f(1) = 0$;
no local minimum. No absolute
or local maximum.

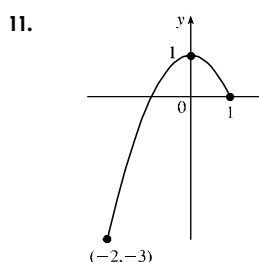
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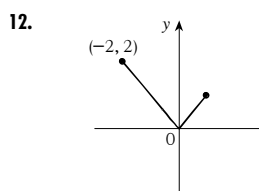
$f(x) = 1 - x^2, 0 \leq x < 1$.
Absolute maximum $f(0) = 1$;
no local maximum. No absolute
or local minimum.



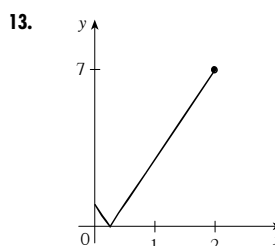
$f(x) = 1 - x^2, 0 \leq x \leq 1$.
Absolute maximum $f(0) = 1$;
no local maximum. Absolute
minimum $f(1) = 0$; no local
minimum.



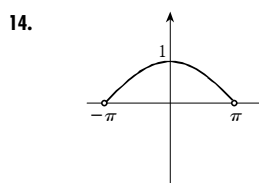
$f(x) = 1 - x^2, -2 \leq x \leq 1$.
Absolute and local maximum
 $f(0) = 1$. Absolute minimum
 $f(-2) = -3$; no local
minimum.



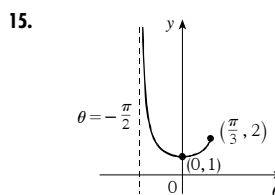
$f(x) = |x|, -2 \leq x \leq 1$.
Absolute maximum $f(-2) = 2$;
no local maximum. Absolute and
local minimum $f(0) = 0$.



$f(x) = |4x - 1|, 0 \leq x \leq 2$.
Absolute maximum $f(2) = 7$;
no local maximum. Absolute and
local minimum $f(\frac{1}{4}) = 0$ since
 $|4x - 1| \geq 0$ for all x .

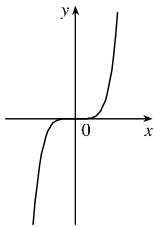


$f(\theta) = \cos(\theta/2)$,
 $-\pi < \theta < \pi$. Local and
absolute maximum $f(0) = 1$.
No local or absolute minimum.



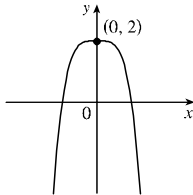
$f(\theta) = \sec \theta, -\frac{\pi}{2} < \theta \leq \frac{\pi}{3}$.
Local and absolute minimum
 $f(0) = 1$. No absolute or local
maximum.

16.



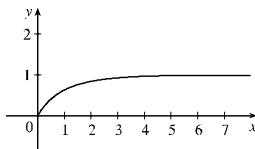
$f(x) = x^5$. No maximum or minimum.

17.



$f(x) = 2 - x^4$. Local and absolute maximum $f(0) = 2$. No local or absolute minimum.

18.



$f(x) = 1 - e^{-x}$, $x \geq 0$. Absolute minimum $f(0) = 0$; no local minimum. No absolute or local maximum.

19. $f(x) = 2x - 3x^2 \Rightarrow f'(x) = 2 - 6x = 0 \Leftrightarrow x = \frac{1}{3}$. So the critical number is $\frac{1}{3}$.

20. $f(x) = 5 + 8x \Rightarrow f'(x) = 8 \neq 0$. No critical number.

21. $f(x) = x^3 - 3x + 1 \Rightarrow f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x+1)(x-1)$. So the critical numbers are ± 1 .

22. $f(t) = t^3 + 6t^2 + 3t - 1 \Rightarrow f'(t) = 3t^2 + 12t + 3 = 3(t^2 + 4t + 1)$. By the quadratic formula, solutions are $t = \frac{-4 \pm \sqrt{12}}{2} = -2 \pm \sqrt{3}$. Critical numbers are $t = -2 \pm \sqrt{3}$.

23. $g(x) = \sqrt[9]{x} = x^{1/9} \Rightarrow g'(x) = \frac{1}{9}x^{-8/9} = \frac{1}{9\sqrt[9]{x^8}} \neq 0$, but $g'(0)$ does not exist, so $x = 0$ is a critical number.

24. $g(x) = |x + 1| \Rightarrow g'(x) = 1$ if $x > -1$, $g'(x) = -1$ if $x < -1$, but $g'(-1)$ does not exist, so $x = -1$ is a critical number.

25. $f(x) = 5 + 6x - 2x^3 \Rightarrow f'(x) = 6 - 6x^2 = 6(1+x)(1-x)$. $f'(x) = 0 \Rightarrow x = \pm 1$, so ± 1 are the critical numbers.

26. $f(t) = 2t^3 + 3t^2 + 6t + 4 \Rightarrow f'(t) = 6t^2 + 6t + 6$. But $t^2 + t + 1 = 0$ has no real solution since $b^2 - 4ac = 1 - 4(1)(1) = -3 < 0$. No critical number.

27. $f(x) = 4x^3 - 9x^2 - 12x + 3 \Rightarrow f'(x) = 12x^2 - 18x - 12 = 6(2x^2 - 3x - 2) = 6(2x+1)(x-2)$
 $f'(x) = 0 \Rightarrow x = -\frac{1}{2}, 2$, so the critical numbers are $x = -\frac{1}{2}, 2$.

28. $s(t) = 2t^3 + 3t^2 - 6t + 4 \Rightarrow s'(t) = 6t^2 + 6t - 6 = 6(t^2 + t - 1)$. By the quadratic formula, the critical numbers are $t = \frac{-1 \pm \sqrt{5}}{2}$.

29. $s(t) = t^4 + 4t^3 + 2t^2 \Rightarrow s'(t) = 4t^3 + 12t^2 + 4t = 4t(t^2 + 3t + 1) = 0$ when $t = 0$ or $t^2 + 3t + 1 = 0$. By the quadratic formula, the critical numbers are $t = 0, \frac{-3 \pm \sqrt{5}}{2}$.

30. $f(r) = \frac{r}{r^2 + 1} \Rightarrow f'(r) = \frac{(r^2 + 1)1 - r(2r)}{(r^2 + 1)^2} = \frac{-r^2 + 1}{(r^2 + 1)^2} = 0 \Leftrightarrow r^2 = 1 \Leftrightarrow r = \pm 1$, so these are the critical numbers. Note that $f'(r)$ always exists since $r^2 + 1 \neq 0$.

31. $f(\theta) = \sin^2(2\theta) \Rightarrow f'(\theta) = 2\sin(2\theta)\cos(2\theta)(2) = 2(2\sin 2\theta \cos 2\theta) = 2[\sin(2 \cdot 2\theta)] = 2\sin 4\theta = 0$
 $f(\theta) = 0 \Leftrightarrow \sin 4\theta = 0 \Leftrightarrow 4\theta = n\pi$, n an integer. So $\theta = n\pi/4$ are the critical numbers.

32. $g(\theta) = \theta + \sin \theta \Rightarrow g'(\theta) = 1 + \cos \theta = 0 \Leftrightarrow \cos \theta = -1$. The critical numbers are $\theta = \pi + 2n\pi = (2n+1)\pi$, n an integer.

33. $V(x) = x\sqrt{x-2} \Rightarrow V'(x) = \sqrt{x-2} + \frac{x}{2\sqrt{x-2}} \Rightarrow V'(-2)$ does not exist. For $x > 2$ [the domain of $V'(x)$], $V'(x) > 0$, so 2 is the only critical number.

34. $T(x) = x^2(2x-1)^{2/3} \Rightarrow T'(x) = 2x(2x-1)^{2/3} + x^2(\frac{2}{3})(2x-1)^{-1/3}(2)$. So $T'(\frac{1}{2})$ does not exist.
 $T'(x) = 2x(2x-1)^{-1/3}(2x-1 + \frac{2}{3}x) = 2x(2x-1)^{-1/3}(\frac{8}{3}x-1) = 0 \Leftrightarrow x = 0$ or $x = \frac{3}{8}$. So the critical numbers are $x = 0, \frac{3}{8}$, and $\frac{1}{2}$.

35. $f(x) = x^2 - 2x + 2, [0, 3]$. $f'(x) = 2x - 2 = 0 \Leftrightarrow x = 1$. $f(0) = 2, f(1) = 1, f(3) = 5$. So $f(3) = 5$ is the absolute maximum and $f(1) = 1$ is the absolute minimum.

36. $f(x) = 1 - 2x - x^2, [-4, 1]$. $f'(x) = -2 - 2x = 0 \Leftrightarrow x = -1$. $f(-4) = -7, f(-1) = 2, f(1) = -2$. So $f(-4) = -7$ is the absolute minimum, $f(-1) = 2$ is the absolute maximum.

37. $f(x) = x^3 - 12x + 1, [-3, 5]$.
 $f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x+2)(x-2) = 0 \Leftrightarrow x = \pm 2$. $f(-3) = 10, f(-2) = 17, f(2) = -15, f(5) = 66$. So $f(2) = -15$ is the absolute minimum and $f(5) = 66$ is the absolute maximum.

38. $f(x) = 4x^3 - 15x^2 + 12x + 7, [0, 3]$.

$f'(x) = 12x^2 - 30x + 12 = 6(2x - 1)(x - 2) = 0 \Leftrightarrow x = \frac{1}{2}, 2$. $f(0) = 7, f(\frac{1}{2}) = \frac{39}{4}, f(2) = 3, f(3) = 16$. So $f(3) = 16$ is the absolute maximum and $f(2) = 3$ the absolute minimum.

39. $f(x) = 2x^3 + 3x^2 + 4, [-2, 1]$.

$f'(x) = 6x^2 + 6x = 6x(x + 1) = 0 \Leftrightarrow x = -1, 0$.
 $f(-2) = 0, f(-1) = 5, f(0) = 4, f(1) = 9$. So $f(1) = 9$ is the absolute maximum and $f(-2) = 0$ is the absolute minimum.

40. $f(x) = 18x + 15x^2 - 4x^3, [-3, 4]$.

$f'(x) = 18 + 30x - 12x^2 = 6(3 - x)(1 + 2x) = 0 \Leftrightarrow x = 3, -\frac{1}{2}$. $f(-3) = 189, f(-\frac{1}{2}) = -\frac{19}{4}, f(3) = 81, f(4) = 56$. So $f(-3) = 189$ is the absolute maximum and $f(-\frac{1}{2}) = -\frac{19}{4}$ is the absolute minimum.

41. $f(x) = x^4 - 4x^2 + 2, [-3, 2]$.

$f'(x) = 4x^3 - 8x = 4x(x^2 - 2) = 0 \Leftrightarrow x = 0, \pm\sqrt{2}$.
 $f(-3) = 47, f(-\sqrt{2}) = -2, f(0) = 2, f(\sqrt{2}) = -2, f(2) = 2$, so $f(\pm\sqrt{2}) = -2$ is the absolute minimum and $f(-3) = 47$ is the absolute maximum.

42. $f(x) = 3x^5 - 5x^3 - 1, [-2, 2]$.

$f'(x) = 15x^4 - 15x^2 = 15x^2(x + 1)(x - 1) = 0 \Leftrightarrow x = -1, 0, 1$. $f(-2) = -57, f(-1) = 1, f(0) = -1, f(1) = -3, f(2) = 55$. So $f(-2) = -57$ is the absolute minimum and $f(2) = 55$ is the absolute maximum.

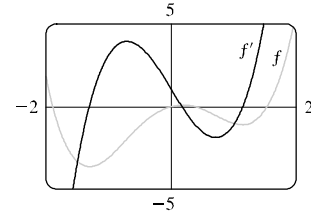
43. $f(x) = x^2 + \frac{2}{x}, [\frac{1}{2}, 2]$. $f'(x) = 2x - \frac{2}{x^2} = 2\frac{x^3 - 1}{x^2} = 0 \Leftrightarrow x^3 - 1 = 0 \Leftrightarrow (x - 1)(x^2 + x + 1) = 0$, but $x^2 + x + 1 \neq 0$, so $x = 1$. The denominator is 0 at $x = 0$, but not in the desired interval. $f(\frac{1}{2}) = \frac{17}{4}, f(1) = 3, f(2) = 5$. So $f(1) = 3$ is the absolute minimum and $f(2) = 5$ is the absolute maximum.

44. $f(x) = \sqrt{9 - x^2}, [-1, 2]$. $f'(x) = -x/\sqrt{9 - x^2} = 0 \Leftrightarrow x = 0$. $f(-1) = 2\sqrt{2}, f(0) = 3, f(2) = \sqrt{5}$. So $f(2) = \sqrt{5}$ is the absolute minimum and $f(0) = 3$ is the absolute maximum.

45. $f(x) = \frac{x}{x+1}, [1, 2]$.

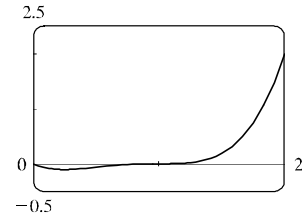
$f'(x) = \frac{(x+1) - x}{(x+1)^2} = \frac{1}{(x+1)^2} \neq 0 \Rightarrow$ no critical number. $f(1) = \frac{1}{2}$ and $f(2) = \frac{2}{3}$, so $f(1) = \frac{1}{2}$ is the absolute minimum and $f(2) = \frac{2}{3}$ is the absolute maximum.

46.



We see that $f'(x) = 0$ at about $x = -1.3, 0.2$, and 1.1 . Since f' exists everywhere, these are the only critical numbers.

47. (a)



From the graph, it appears that the absolute maximum value is $f(2) = 2$, and that the absolute minimum value is about $f(0.25) = -0.11$.

(b) $f(x) = x^4 - 3x^3 + 3x^2 - x \Rightarrow f'(x) = 4x^3 - 9x^2 + 6x - 1 = (4x - 1)(x - 1)^2$.
 So $f'(x) = 0 \Rightarrow x = \frac{1}{4}$ or $x = 1$.
 Now $f(1) = 1^4 - 3 \cdot 1^3 + 3 \cdot 1^2 - 1 = 0$ (neither maximum nor minimum) and
 $f(\frac{1}{4}) = (\frac{1}{4})^4 - 3(\frac{1}{4})^3 + 3(\frac{1}{4})^2 - \frac{1}{4} = -\frac{27}{256}$ (minimum). At the right endpoint we have
 $f(2) = 2^4 - 3 \cdot 2^3 + 3 \cdot 2^2 - 2 = 2$ (maximum).

48. $f(x) = x^5$. $f'(x) = 5x^4 \Rightarrow f'(0) = 0$ so 0 is a critical number. But $f(0) = 0$ and f takes both positive and negative values in any open interval containing 0, so f has neither a local maximum nor a local minimum at 0.

49.

