3.6 Implicit Differentiation

A Click here for answers.

S Click here for solutions.

1–5

(a) Find y' by implicit differentiation.

(b) Solve the equation explicitly for y and differentiate to get y' in terms of x.

(c) Check that your solutions to parts (a) and (b) are consistent by substituting the expression for y into your solution for part (a).

1. $x^2 + 3x + xy = 5$

2. $\dfrac{x^2}{2} + \dfrac{y^2}{4} = 1$

3. $2y^2 + xy = x^2 + 3$

4. $\dfrac{1}{x} + \dfrac{1}{y} = 3$

5. $x^2 + xy - y^2 = 3$

6–16 Find dy/dx by implicit differentiation.

6. $y^5 + 3x^2y^2 + 5x^4 = 12$

7. $x^4 + y^4 = 16$

8. $\dfrac{y}{x - y} = x^2 + 1$

9. $x\sqrt{1 + y} + y\sqrt{1 + 2x} = 2x$

10. $2xy = (x^2 + y^2)^{3/2}$

11. $x^2 = \dfrac{y^2}{y^2 - 1}$

12. $\sqrt{x + y} + \sqrt{xy} = 6$

13. $\sqrt{1 + x^2y^2} = 2xy$

14. $x \sin y + \cos 2y = \cos y$

15. $\cos(x - y) = xe^x$

16. $x \cos y + y \cos x = 1$

17. If $x[f(x)]^3 + xf(x) = 6$ and $f(3) = 1$, find $f'(3)$.

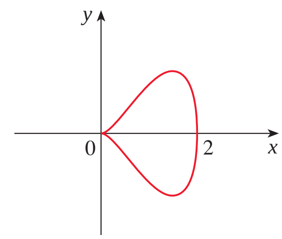
18. If $[g(x)]^2 + 12x = x^2g(x)$ and $g(4) = 12$, find $g'(4)$.

19–22 Find an equation of the tangent line to the curve at the given point.

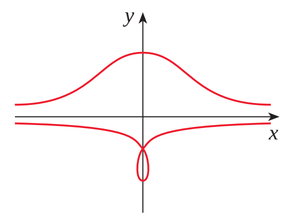
19. $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1, \quad \left(-5, \frac{9}{4}\right)$ (hyperbola)

20. $\dfrac{x^2}{9} + \dfrac{y^2}{36} = 1, \quad \left(-1, 4\sqrt{2}\right)$ (ellipse)

21. $y^2 = x^3(2 - x)$
$(1, 1)$
(piriform)



22. $x^2y^2 = (y + 1)^2(4 - y^2)$
$(0, -2)$
(conchoid of Nicomedes)



23–27 Find the derivative of the function. Simplify where possible.

23. $y = \sin^{-1}(x^2)$

24. $y = (\sin^{-1}x)^2$

25. $y = \tan^{-1}(e^x)$

26. $g(t) = \sin^{-1}(4/t)$

27. $y = x^2 \cot^{-1}(3x)$

Answers

E Click here for exercises.

S Click here for solutions.

1. (a) $y' = -(2x + y + 3)/x$

(b) $y = \frac{5}{x} - x - 3,\ y' = -\frac{5}{x^2} - 1$

2. (a) $y' = -\frac{2x}{y}$

(b) $y = \pm\sqrt{4 - 2x^2},\ y' = \mp\frac{2x}{\sqrt{4 - 2x^2}}$

3. (a) $y' = \frac{2x - y}{x + 4y}$

(b) $y = \frac{1}{4}\left(-x \pm \sqrt{9x^2 + 24}\right),\ y' = \frac{1}{4}\left(-1 \pm \frac{9x}{\sqrt{9x^2 + 24}}\right)$

4. (a) $y' = -\frac{y^2}{x^2}$

(b) $y = \frac{x}{3x - 1},\ y' = -\frac{1}{(3x - 1)^2}$

5. (a) $y' = \frac{2x + y}{2y - x}$

(b) $y = \frac{1}{2}\left(x \pm \sqrt{5x^2 - 12}\right),\ y' = \frac{1}{2}\left(1 \pm \frac{5x}{\sqrt{5x^2 - 12}}\right)$

6. $-\frac{20x^3 + 6xy^2}{5y^4 + 6x^2y}$ **7.** $-\frac{x^3}{y^3}$

8. $\frac{y}{x} + 2(x - y)^2$ or $\frac{3x^2 + 1 - 2xy}{x^2 + 2}$

9. $\frac{2 - \sqrt{1 + y} - y/\sqrt{1 + 2x}}{\sqrt{1 + 2x} + x/\left(2\sqrt{1 + y}\right)}$ **10.** $\frac{3x(x^2 + y^2)^{1/2} - 2y}{2x - 3y(x^2 + y^2)^{1/2}}$

11. $-\frac{x(y^2 - 1)^2}{y}$ **12.** $-\frac{\sqrt{xy} + y\sqrt{x + y}}{\sqrt{xy} + x\sqrt{x + y}}$ **13.** $-\frac{y}{x}$

14. $\frac{\sin y}{2\sin 2y - x\cos y - \sin y}$ **15.** $1 + \frac{e^x(1 + x)}{\sin(x - y)}$

16. $\frac{y\sin x - \cos y}{\cos x - x\sin y}$ **17.** $-\frac{1}{6}$ **18.** $\frac{21}{2}$ **19.** $y = -\frac{5}{4}x - 4$

20. $y = \frac{1}{\sqrt{2}}(x + 9)$ **21.** $y = x$ **22.** $y = -2$

23. $y' = \frac{2x}{\sqrt{1 - x^4}}$ **24.** $y' = \frac{2\sin^{-1}x}{\sqrt{1 - x^2}}$

25. $y' = \frac{e^x}{1 + e^{2x}}$ **26.** $y' = -\frac{4}{\sqrt{t^4 - 16t^2}}$

27. $y' = 2x\cot^{-1}(3x) - \frac{3x^2}{1 + 9x^2}$

Solutions

E Click here for exercises.

A Click here for answers.

1. (a) $x^2 + 3x + xy = 5 \Rightarrow 2x + 3 + y + xy' = 0 \Rightarrow$
$y' = -\dfrac{2x + y + 3}{x}$

(b) $x^2 + 3x + xy = 5 \Rightarrow$
$y = \dfrac{5 - x^2 - 3x}{x} = \dfrac{5}{x} - x - 3 \Rightarrow y' = -\dfrac{5}{x^2} - 1$

(c) $y' = -\dfrac{2x + y + 3}{x}$
$= \dfrac{-2x - 3 - (-3 - x + 5/x)}{x} = -1 - \dfrac{5}{x^2}$

2. (a) $\dfrac{x^2}{2} + \dfrac{y^2}{4} = 1 \Rightarrow x + \dfrac{y}{2}y' = 0 \Rightarrow y' = -\dfrac{2x}{y}$

(b) $\dfrac{y^2}{4} = 1 - \dfrac{x^2}{2} \Rightarrow y^2 = 4 - 2x^2$
$\Rightarrow y = \pm\sqrt{4 - 2x^2} \Rightarrow$
$y' = \pm\dfrac{1}{2\sqrt{4 - 2x^2}}(-4x) = \mp\dfrac{2x}{\sqrt{4 - 2x^2}}$

(c) $y' = \dfrac{-2x}{y} = \dfrac{-2x}{\pm\sqrt{4 - 2x^2}} = \mp\dfrac{2x}{\sqrt{4 - 2x^2}}$

3. (a) $2y^2 + xy = x^2 + 3 \Rightarrow 4yy' + y + xy' = 2x \Rightarrow$
$y' = \dfrac{2x - y}{x + 4y}$

(b) Use the quadratic formula:
$2y^2 + xy - (x^2 + 3) = 0 \Rightarrow$
$y = \dfrac{-x \pm \sqrt{x^2 + 8(x^2 + 3)}}{4} = \dfrac{-x \pm \sqrt{9x^2 + 24}}{4}$
$\Rightarrow y' = \dfrac{1}{4}\left(-1 \pm \dfrac{9x}{\sqrt{9x^2 + 24}}\right)$

(c) $y' = \dfrac{2x - y}{x + 4y} = \dfrac{2x - \frac{1}{4}\left(-x \pm \sqrt{9x^2 + 24}\right)}{x + \left(-x \pm \sqrt{9x^2 + 24}\right)}$
$= \dfrac{1}{4}\left(-1 \pm \dfrac{9x}{\sqrt{9x^2 + 24}}\right)$

4. (a) $\dfrac{1}{x} + \dfrac{1}{y} = 3 \Rightarrow -\dfrac{1}{x^2} - \dfrac{1}{y^2}y' = 0 \Rightarrow y' = -\dfrac{y^2}{x^2}$

(b) $\dfrac{1}{y} = 3 - \dfrac{1}{x} = \dfrac{3x - 1}{x} \Rightarrow y = \dfrac{x}{3x - 1} \Rightarrow$
$y' = \dfrac{(3x - 1) - (x)(3)}{(3x - 1)^2} = -\dfrac{1}{(3x - 1)^2}$

(c) $y' = -\dfrac{y^2}{x^2} = -\dfrac{x^2/(3x - 1)^2}{x^2} = -\dfrac{1}{(3x - 1)^2}$

5. (a) $x^2 + xy - y^2 = 3 \Rightarrow 2x + y + xy' - 2yy' = 0 \Rightarrow$
$y' = \dfrac{2x + y}{2y - x}$

(b) Use the quadratic formula: $y^2 - xs + (3 - x^2) = 0 \Rightarrow$
$y = \frac{1}{2}\left[x \pm \sqrt{x^2 - 4(3 - x^2)}\right] = \frac{1}{2}\left(x \pm \sqrt{5x^2 - 12}\right)$
$\Rightarrow y' = \dfrac{1}{2}\left(1 \pm \dfrac{5x}{\sqrt{5x^2 - 12}}\right)$

(c) $y' = \dfrac{2x + y}{2y - x} = \dfrac{2x + \frac{1}{2}\left(x \pm \sqrt{5x^2 - 12}\right)}{x \pm \sqrt{5x^2 - 12} - x}$
$= \dfrac{1}{2}\left(1 \pm \dfrac{5x}{\sqrt{5x^2 - 12}}\right)$

6. $y^5 + 3x^2y^2 + 5x^4 = 12 \Rightarrow$
$5y^4y' + 6xy^2 + 6x^2yy' + 20x^3 = 0 \Rightarrow$
$y' = -\dfrac{20x^3 + 6xy^2}{5y^4 + 6x^2y}$

7. $x^4 + y^4 = 16 \Rightarrow 4x^3 + 4y^3y' = 0 \Rightarrow y' = -\dfrac{x^3}{y^3}$

8. $\dfrac{y}{x - y} = x^2 + 1 \Rightarrow$
$2x = \dfrac{(x - y)\,y' - y\,(1 - y')}{(x - y)^2} = \dfrac{xy' - y}{(x - y)^2} \Rightarrow$
$y' = \dfrac{y}{x} + 2\,(x - y)^2.$

Another Method: Write the equation as
$y = (x - y)\,(x^2 + 1) = x^3 + x - yx^2 - y.$ Then
$y' = \dfrac{3x^2 + 1 - 2xy}{x^2 + 2}.$

9. $x\sqrt{1 + y} + y\sqrt{1 + 2x} = 2x \Rightarrow$
$\sqrt{1 + y} + x\dfrac{1}{2\sqrt{1 + y}}y' + y'\sqrt{1 + 2x} + y\dfrac{2}{2\sqrt{1 + 2x}} = 2$
$\Rightarrow y' = \dfrac{2 - \sqrt{1 + y} - \dfrac{y}{\sqrt{1 + 2x}}}{\sqrt{1 + 2x} + \dfrac{x}{2\sqrt{1 + y}}}$

10. $2xy = \left(x^2 + y^2\right)^{3/2} \Rightarrow$
$2y + 2xy' = \frac{3}{2}\left(x^2 + y^2\right)^{1/2}(2x + 2yy') \Rightarrow$
$y' = \dfrac{3x\left(x^2 + y^2\right)^{1/2} - 2y}{2x - 3y\,(x^2 + y^2)^{1/2}}$

11. $x^2 = \dfrac{y^2}{y^2-1} \quad\Rightarrow$

$$2x = \frac{(y^2-1)\,2yy' - y^2\,(2yy')}{(y^2-1)^2} = -\frac{2yy'}{(y^2-1)^2} \quad\Rightarrow$$

$$y' = -\frac{x\left(y^2-1\right)^2}{y}.$$

Another Method: Write the equation as $x^2\left(y^2-1\right) = y^2$. This gives $y' = \dfrac{x - xy^2}{x^2y - y}$.

12. $\sqrt{x+y} + \sqrt{xy} = 6 \quad\Rightarrow$

$$\tfrac{1}{2}(x+y)^{-1/2}(1+y') + \tfrac{1}{2}(xy)^{-1/2}(y+xy') = 0 \Rightarrow$$

$$(x+y)^{-1/2} + (x+y)^{-1/2}y' + (xy)^{-1/2}y + (xy)^{-1/2}xy' = 0$$

$\Rightarrow$

$$y' = -\frac{(x+y)^{-1/2} + (xy)^{-1/2}y}{(x+y)^{-1/2} + (xy)^{-1/2}x} \cdot \frac{(x+y)^{1/2}(xy)^{1/2}}{(x+y)^{1/2}(xy)^{1/2}} = -\frac{\sqrt{xy} + y\sqrt{x+y}}{\sqrt{xy} + x\sqrt{x+y}}$$

13. $\sqrt{1+x^2y^2} = 2xy \quad\Rightarrow$

$$\tfrac{1}{2}\left(1+x^2y^2\right)^{-1/2}\left(x^2\cdot 2yy' + y^2\cdot 2x\right) = 2\,(xy' + y\cdot 1)$$

$$\Rightarrow \quad \frac{2x^2y}{2\sqrt{1+x^2y^2}}\,y' + \frac{2xy^2}{2\sqrt{1+x^2y^2}} = 2xy' + 2y \quad\Rightarrow$$

$$y'\left(\frac{x^2y}{\sqrt{1+x^2y^2}} - 2x\right) = 2y - \frac{xy^2}{\sqrt{1+x^2y^2}} \quad\Rightarrow$$

$$y'\left(\frac{x^2y - 2x\sqrt{1+x^2y^2}}{\sqrt{1+x^2y^2}}\right) = \frac{2y\sqrt{1+x^2y^2} - xy^2}{\sqrt{1+x^2y^2}} \quad\Rightarrow$$

$$y' = \frac{2y\sqrt{1+x^2y^2} - xy^2}{x^2y - 2x\sqrt{1+x^2y^2}} = \frac{y\left(2\sqrt{1+x^2y^2} - xy\right)}{x\left(xy - 2\sqrt{1+x^2y^2}\right)} = -\frac{y}{x}$$

Another Method: Since $1 + x^2y^2$ is positive, we can square both sides first and then differentiate implicitly.

14. $x\sin y + \cos 2y = \cos y \quad\Rightarrow$

$$\sin y + (x\cos y)\,y' - (2\sin 2y)\,y' = (-\sin y)\,y' \quad\Rightarrow$$

$$\sin y = (2\sin 2y)\,y' - (x\cos y)\,y' - (\sin y)\,y' \quad\Rightarrow$$

$$y' = \frac{\sin y}{2\sin 2y - x\cos y - \sin y}$$

15. $\cos(x-y) = xe^x \quad\Rightarrow$

$$-\sin(x-y)\,(1-y') = xe^x + e^x \quad\Rightarrow$$

$$1 - y' = -\frac{(x+1)\,e^x}{\sin(x-y)} \quad\Rightarrow\quad y' = 1 + \frac{(x+1)\,e^x}{\sin(x-y)}$$

16. $x\cos y + y\cos x = 1 \quad\Rightarrow$

$$\cos y + x(-\sin y)\,y' + y'\cos x - y\sin x = 0 \quad\Rightarrow$$

$$y' = \frac{y\sin x - \cos y}{\cos x - x\sin y}$$

17. $x\,[f(x)]^3 + xf(x) = 6 \quad\Rightarrow$

$$[f(x)]^3 + 3x\,[f(x)]^2 f'(x) + f(x) + xf'(x) = 0 \quad\Rightarrow$$

$$f'(x) = -\frac{[f(x)]^3 + f(x)}{3x\,[f(x)]^2 + x} \quad\Rightarrow$$

$$f'(3) = -\frac{(1)^3 + 1}{3\,(3)\,(1)^2 + 3} = -\frac{1}{6}$$

18. $[g(x)]^2 + 12x = x^2 g(x) \quad\Rightarrow$

$$2g(x)\,g'(x) + 12 = 2xg(x) + x^2 g'(x)$$

$$\Leftrightarrow \quad g'(x) = \frac{2xg(x) - 12}{2g(x) - x^2} \quad\Rightarrow$$

$$g'(4) = \frac{2\,(4)\,(12) - 12}{2\,(12) - (4)^2} = \frac{21}{2}$$

19. $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1 \quad\Rightarrow\quad \dfrac{x}{8} - \dfrac{2yy'}{9} = 0 \quad\Rightarrow\quad y' = \dfrac{9x}{16y}$.

When $x = -5$ and $y = \frac{9}{4}$ we have $y' = \dfrac{9\,(-5)}{16\,(9/4)} = -\dfrac{5}{4}$ so an equation of the tangent is $y - \frac{9}{4} = -\frac{5}{4}(x+5)$ or $y = -\frac{5}{4}x - 4$.

20. $\dfrac{x^2}{9} + \dfrac{y^2}{36} = 1 \quad\Rightarrow\quad \dfrac{2x}{9} + \dfrac{yy'}{18} = 0 \quad\Rightarrow\quad y' = -\dfrac{4x}{y}$.

When $x = -1$ and $y = 4\sqrt{2}$ we have $y' = -\dfrac{4\,(-1)}{4\sqrt{2}} = \dfrac{1}{\sqrt{2}}$ so an equation of the tangent line is $y - 4\sqrt{2} = \frac{1}{\sqrt{2}}(x+1)$ or $y = \frac{1}{\sqrt{2}}(x+9)$.

21. $y^2 = x^3(2-x) = 2x^3 - x^4 \quad\Rightarrow\quad 2yy' = 6x^2 - 4x^3$

$\Rightarrow \quad y' = \dfrac{3x^2 - 2x^3}{y}$. When $x = y = 1$,

$y' = \dfrac{3\,(1)^2 - 2\,(1)^3}{1} = 1$, so an equation of the tangent line is $y - 1 = 1\,(x-1)$ or $y = x$.

22. $x^2y^2 = (y+1)^2\left(4-y^2\right) \quad\Rightarrow$

$$2xy^2 + 2x^2yy' = 2\,(y+1)\,y'\left(4-y^2\right) + (y+1)^2\,(-2yy')$$

$\Rightarrow \quad y' = \dfrac{xy^2}{(y+1)\,(4-y^2) - y\,(y+1)^2 - x^2y} = 0$ when $x = 0$. So an equation of the tangent line at $(0,-2)$ is $y + 2 = 0\,(x-0)$ or $y = -2$.

23. $y = \sin^{-1}\left(x^2\right) \quad\Rightarrow$

$$y' = \frac{1}{\sqrt{1-(x^2)^2}}\,\frac{d}{dx}\left(x^2\right) = \frac{2x}{\sqrt{1-x^4}}$$

24. $y = \left(\sin^{-1}x\right)^2 \quad\Rightarrow\quad y' = 2\left(\sin^{-1}x\right)\dfrac{d}{dx}\left(\sin^{-1}x\right) \quad\Rightarrow$

$$y' = \frac{2\sin^{-1}x}{\sqrt{1-x^2}}$$

25. $y = \tan^{-1}(e^x) \quad \Rightarrow \quad y' = \dfrac{1}{1+(e^x)^2}\dfrac{d}{dx}(e^x) = \dfrac{e^x}{1+e^{2x}}$

26. $g(t) = \sin^{-1}\left(\dfrac{4}{t}\right) \quad \Rightarrow$

$$g'(t) = \frac{1}{\sqrt{1-(4/t)^2}}\left(-\frac{4}{t^2}\right) = -\frac{4}{\sqrt{t^4-16t^2}}$$

27. $y = x^2\cot^{-1}(3x) \quad \Rightarrow$

$$y' = 2x\cot^{-1}(3x) + x^2\left[-\frac{1}{1+(3x)^2}\right](3)$$

$$= 2x\cot^{-1}(3x) - \frac{3x^2}{1+9x^2}$$