3.4 Derivatives of Trigonometric Functions

A Click here for answers.

S Click here for solutions.

1–8 Differentiate.

1. $y = \sin x + \cos x$
2. $y = \cos x - 2 \tan x$
3. $y = e^x \sin x$
4. $y = \dfrac{\tan x}{x}$
5. $y = \dfrac{\sin x}{1 + \cos x}$
6. $y = \dfrac{x}{\sin x + \cos x}$
7. $y = \tan \theta (\sin \theta + \cos \theta)$
8. $y = \csc x \cot x$

9–10 Find an equation of the tangent line to the given curve at the specified point.

9. $y = 2 \sin x, \quad (\pi/6, 1)$
10. $y = \sec x - 2 \cos x, \quad (\pi/3, 1)$

11–22 Find the limit.

11. $\displaystyle\lim_{t \to 0} \frac{\sin 5t}{t}$
12. $\displaystyle\lim_{t \to 0} \frac{\sin 8t}{\sin 9t}$
13. $\displaystyle\lim_{\theta \to 0} \frac{\sin^2 \theta}{\theta}$
14. $\displaystyle\lim_{x \to 0} \frac{\tan x}{4x}$
15. $\displaystyle\lim_{h \to 0} \frac{\sin 5h}{\tan 3h}$
16. $\displaystyle\lim_{x \to 0} \frac{\tan 3x}{3 \tan 2x}$
17. $\displaystyle\lim_{x \to 0} \frac{1 - \cos x}{2x^2}$
18. $\displaystyle\lim_{x \to \pi} \frac{\tan x}{\sin 2x}$
19. $\displaystyle\lim_{x \to 0} \frac{\cos x \sin x - \tan x}{x^2 \sin x}$
20. $\displaystyle\lim_{y \to 0} \left(\lim_{x \to 0} \frac{\cos x \sin y}{x - y} \right)$
21. $\displaystyle\lim_{x \to 0} \frac{\sin(\sin x)}{\sin x}$
22. $\displaystyle\lim_{x \to 0} \frac{\sin(\sin x)}{x}$

Answers

E Click here for exercises.

S Click here for solutions.

1. $dy/dx = \cos x - \sin x$ **2.** $dy/dx = -\sin x - 2\sec^2 x$

3. $dy/dx = e^x(\cos x + \sin x)$

4. $\dfrac{dy}{dx} = \dfrac{x\sec^2 x - \tan x}{x^2}$ **5.** $\dfrac{dy}{dx} = \dfrac{1}{1 + \cos x}$

6. $\dfrac{dy}{dx} = \dfrac{(1 + x)\sin x + (1 - x)\cos x}{1 + \sin 2x}$

7. $y' = \sin\theta - \sin\theta\tan\theta + \sin\theta\sec^2\theta + \sec\theta$

8. $dy/dx = -\csc x\,(\cot^2 x + \csc^2 x)$

9. $y = \sqrt{3}x + 1 - \frac{1}{6}\sqrt{3}\pi$ **10.** $y = 3\sqrt{3}x + 1 - \pi\sqrt{3}$

11. 5 **12.** $\frac{8}{9}$ **13.** 0 **14.** $\frac{1}{4}$ **15.** $\frac{5}{3}$ **16.** $\frac{1}{2}$ **17.** $\frac{1}{4}$

18. $\frac{1}{2}$ **19.** -1 **20.** -1 **21.** 1 **22.** 1

Solutions

E Click here for exercises.

A Click here for answers.

1. $y = \sin x + \cos x \quad \Rightarrow \quad dy/dx = \cos x - \sin x$

2. $y = \cos x - 2\tan x \quad \Rightarrow \quad dy/dx = -\sin x - 2\sec^2 x$

3. $y = e^x \sin x \quad \Rightarrow$
$dy/dx = e^x(\cos x) + (\sin x)e^x = e^x(\cos x + \sin x)$

4. $y = \dfrac{\tan x}{x} \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{x\sec^2 x - \tan x}{x^2}$

5. $y = \dfrac{\sin x}{1+\cos x} \quad \Rightarrow$

$$\frac{dy}{dx} = \frac{(1+\cos x)\cos x - \sin x(-\sin x)}{(1+\cos x)^2} = \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2} = \frac{\cos x + 1}{(1+\cos x)^2} = \frac{1}{1+\cos x}$$

6. $y = \dfrac{x}{\sin x + \cos x} \quad \Rightarrow$

$$\frac{dy}{dx} = \frac{(\sin x + \cos x) - x(\cos x - \sin x)}{(\sin x + \cos x)^2} = \frac{(1+x)\sin x + (1-x)\cos x}{\sin^2 x + \cos^2 x + 2\sin x\cos x} = \frac{(1+x)\sin x + (1-x)\cos x}{1 + \sin 2x}$$

7. $y = \tan\theta(\sin\theta + \cos\theta) \quad \Rightarrow$

$$y' = \tan\theta(\cos\theta - \sin\theta) + (\sin\theta + \cos\theta)\sec^2\theta = \sin\theta - \sin\theta\tan\theta + \sin\theta\sec^2\theta + \sec\theta$$

8. $y = \csc x \cot x \quad \Rightarrow$

$$dy/dx = (-\csc x\,\cot x)\cot x + \csc x(-\csc^2 x) = -\csc x(\cot^2 x + \csc^2 x)$$

9. $y = 2\sin x \quad \Rightarrow \quad y' = 2\cos x \quad \Rightarrow$ the slope of the tangent line at $\left(\frac{\pi}{6}, 1\right)$ is $2\cos\frac{\pi}{6} = 2\cdot\frac{\sqrt{3}}{2} = \sqrt{3}$ and an equation is $y - 1 = \sqrt{3}\left(x - \frac{\pi}{6}\right)$ or $y = \sqrt{3}x + 1 - \frac{\sqrt{3}\pi}{6}$.

10. $y = \sec x - 2\cos x \quad \Rightarrow \quad y' = \sec x\tan x + 2\sin x$ $\Rightarrow$ The slope of the tangent line at $\left(\frac{\pi}{3}, 1\right)$ is $\sec\frac{\pi}{3}\tan\frac{\pi}{3} + 2\sin\frac{\pi}{3} = 2\sqrt{3} + 2\cdot\frac{\sqrt{3}}{2} = 3\sqrt{3}$ and an equation is $y - 1 = 3\sqrt{3}\left(x - \frac{\pi}{3}\right)$ or $y = 3\sqrt{3}x + 1 - \pi\sqrt{3}$.

11. $\displaystyle\lim_{t\to 0}\frac{\sin 5t}{t} = \lim_{t\to 0}\frac{5\sin 5t}{5t} = 5\lim_{t\to 0}\frac{\sin 5t}{5t} = 5\cdot 1 = 5$

12. $\displaystyle\lim_{t\to 0}\frac{\sin 8t}{\sin 9t} = \lim_{t\to 0}\frac{8\left(\frac{\sin 8t}{8t}\right)}{9\left(\frac{\sin 9t}{9t}\right)} = \frac{8\lim_{t\to 0}\frac{\sin 8t}{8t}}{9\lim_{t\to 0}\frac{\sin 9t}{9t}} = \frac{8\cdot 1}{9\cdot 1} = \frac{8}{9}$

13. $\displaystyle\lim_{\theta\to 0}\frac{\sin^2\theta}{\theta} = \lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)\sin\theta = \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\lim_{\theta\to 0}\sin\theta = 1\cdot 0 = 0$

14. $\displaystyle\lim_{x\to 0}\frac{\tan x}{4x} = \lim_{x\to 0}\frac{1}{4}\frac{\sin x}{x}\cdot\frac{1}{\cos x} = \frac{1}{4}\lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}\frac{1}{\cos x} = \frac{1}{4}\cdot 1\cdot 1 = \frac{1}{4}$

15. $\displaystyle\lim_{h\to 0}\frac{\sin 5h}{\tan 3h} = \lim_{h\to 0}\frac{5\frac{\sin 5h}{5h}}{3\frac{\tan 3h}{3h}} = \frac{5}{3}\frac{\lim_{h\to 0}\frac{\sin 5h}{5h}}{\lim_{h\to 0}\frac{\sin 3h}{3h}\cdot\lim_{h\to 0}\frac{1}{\cos 3h}} = \frac{5}{3}\cdot\frac{1}{1\cdot 1} = \frac{5}{3}$

16. $\displaystyle\lim_{x\to 0}\frac{\tan 3x}{3\tan 2x} = \lim_{x\to 0}\frac{\frac{\tan 3x}{3x}}{2\frac{\tan 2x}{2x}} = \frac{1}{2}\frac{\lim_{x\to 0}\frac{\sin 3x}{3x}\cdot\frac{1}{\cos 3x}}{\lim_{x\to 0}\frac{\sin 2x}{2x}\cdot\lim_{x\to 0}\frac{1}{\cos 2x}} = \frac{1}{2}\cdot\frac{1\cdot 1}{1\cdot 1} = \frac{1}{2}$

17. Using the identity $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$, or $1 - \cos x = 2\sin^2(x/2)$, we have

$$\lim_{x\to 0}\frac{1-\cos x}{2x^2} = \lim_{x\to 0}\frac{2\sin^2(x/2)}{2x^2} = \frac{1}{4}\left[\lim_{x\to 0}\frac{\sin(x/2)}{x/2}\right]^2 = \frac{1}{4}(1)^2 = \frac{1}{4}$$

Another Method: Multiply numerator and denominator by $1 + \cos x$.

18. $\displaystyle\lim_{x\to\pi}\frac{\tan x}{\sin 2x} = \lim_{x\to\pi}\frac{\sin x}{\cos x(2\sin x\cos x)} = \lim_{x\to\pi}\frac{1}{2\cos^2 x} = \frac{1}{2(-1)^2} = \frac{1}{2}$

19. $$\lim_{x\to 0}\frac{\cos x\sin x-\tan x}{x^2\sin x}=\lim_{x\to 0}\frac{\cos x\sin x-\sin x/\cos x}{x^2\sin x}$$
$$=\lim_{x\to 0}\frac{\cos^2 x\sin x-\sin x}{x^2\sin x\cos x}$$
$$=\lim_{x\to 0}\frac{\cos^2 x-1}{x^2\cos x}$$
$$=\lim_{x\to 0}\left(\frac{-\sin^2 x}{x^2}\right)\frac{1}{\cos x}$$
$$=-\left[\lim_{x\to 0}\frac{\sin x}{x}\right]^2\left[\lim_{x\to 0}\frac{1}{\cos x}\right]$$
$$=-1$$

20. $$\lim_{y\to 0}\left(\lim_{x\to 0}\frac{\cos x\sin y}{x-y}\right)=\lim_{y\to 0}\left(\frac{1\cdot\sin y}{0-y}\right)$$
$$=\lim_{y\to 0}\left(-\frac{\sin y}{y}\right)=-1$$

Note that

$$\lim_{y\to 0}\left(\lim_{x\to 0}\frac{\cos x\sin y}{x-y}\right)\neq\lim_{x\to 0}\left(\lim_{y\to 0}\frac{\cos x\sin y}{x-y}\right)$$
$$=\lim_{x\to 0}\left(\frac{0}{x}\right)=0$$

21. $\displaystyle\lim_{x\to 0}\frac{\sin(\sin x)}{\sin x}=\lim_{\sin x\to 0}\frac{\sin(\sin x)}{\sin x}$ since as $x\to 0$, $\sin x\to 0$. So we make the substitution $y=\sin x$, and see that $\displaystyle\lim_{x\to 0}\frac{\sin(\sin x)}{\sin x}=\lim_{y\to 0}\frac{\sin y}{y}=1$.

22. $$\lim_{x\to 0}\frac{\sin(\sin x)}{x}=\lim_{x\to 0}\frac{\sin(\sin x)}{\sin x}\cdot\frac{\sin x}{x}$$
$$=\left[\lim_{x\to 0}\frac{\sin(\sin x)}{\sin x}\right]\left[\lim_{x\to 0}\frac{\sin x}{x}\right]$$
$$=1\cdot 1=1$$

See Exercise 21 for a proof that $\displaystyle\lim_{x\to 0}\frac{\sin(\sin x)}{\sin x}=1$.