2.5 Limits Involving Infinity

A Click here for answers.

1–3 Find the limit.

1. $\lim_{x\to 3} \dfrac{1}{(x-3)^8}$

2. $\lim_{x\to \pi^-} \csc x$

3. $\lim_{x\to 2^+} \dfrac{x-1}{x^2(x+2)}$

4–11 Evaluate the limit and justify each step by indicating the appropriate properties of limits.

4. $\lim_{x\to\infty} \dfrac{1}{x\sqrt{x}}$

5. $\lim_{x\to\infty} \dfrac{5+2x}{3-x}$

6. $\lim_{x\to\infty} \dfrac{x+4}{x^2-2x+5}$

7. $\lim_{t\to\infty} \dfrac{7t^3+4t}{2t^3-t^2+3}$

8. $\lim_{x\to-\infty} \dfrac{(1-x)(2+x)}{(1+2x)(2-3x)}$

9. $\lim_{x\to\infty} \sqrt{\dfrac{2x^2-1}{x+8x^2}}$

10. $\lim_{x\to\infty} \dfrac{1}{3+\sqrt{x}}$

11. $\lim_{x\to\infty} \dfrac{\sin^2 x}{x^2}$

12–29 Find the limit.

12. $\lim_{r\to\infty} \dfrac{r^4-r^2+1}{r^5+r^3-r}$

13. $\lim_{t\to-\infty} \dfrac{6t^2+5t}{(1-t)(2t-3)}$

14. $\lim_{x\to\infty} \dfrac{\sqrt{1+4x^2}}{4+x}$

15. $\lim_{x\to-\infty} \dfrac{\sqrt{x^2+4x}}{4x+1}$

S Click here for solutions.

16. $\lim_{x\to\infty} \dfrac{1-\sqrt{x}}{1+\sqrt{x}}$

17. $\lim_{x\to\infty} \left(\sqrt{x^2+3x+1}-x\right)$

18. $\lim_{x\to\infty} \left(\sqrt{x^2+1}-\sqrt{x^2-1}\right)$

19. $\lim_{x\to\infty} \left(\sqrt{1+x}-\sqrt{x}\right)$

20. $\lim_{x\to\infty} \left(\sqrt[3]{1+x}-\sqrt[3]{x}\right)$

21. $\lim_{x\to-\infty} \left(\sqrt{x^2+x+1}+x\right)$

22. $\lim_{x\to\infty} \left(x+\sqrt{x}\right)$

23. $\lim_{x\to-\infty} (x^3-5x^2)$

24. $\lim_{x\to\infty} \dfrac{x^7-1}{x^6+1}$

25. $\lim_{x\to\infty} e^{-x^2}$

26. $\lim_{x\to\infty} (x^2-x^4)$

27. $\lim_{x\to\infty} \dfrac{x^3-1}{x^4+1}$

28. $\lim_{x\to\infty} \dfrac{\sqrt{x}+3}{x+3}$

29. $\lim_{x\to\infty} \dfrac{x}{\sqrt{x-1}}$

30. Guess the value of the limit

$$\lim_{x\to\infty} x^2 \sin\frac{5}{x^2}$$

by evaluating $f(x) = x^2\sin(5/x^2)$ for $x = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 50$, and 100. Then confirm your guess by evaluating this limit exactly.

Answers

E Click here for exercises.

S Click here for solutions.

1. ∞ **2.** ∞ **3.** $-\infty$ **4.** 0 **5.** -2 **6.** 0 **7.** $\frac{7}{2}$
8. $\frac{1}{6}$ **9.** $\frac{1}{2}$ **10.** 0 **11.** 0 **12.** 0 **13.** -3 **14.** 2
15. $-\frac{1}{4}$ **16.** -1 **17.** $\frac{3}{2}$ **18.** 0 **19.** 0 **20.** 0
21. $-\frac{1}{2}$ **22.** ∞ **23.** $-\infty$ **24.** ∞ **25.** 0 **26.** $-\infty$
27. 0 **28.** 0 **29.** ∞ **30.** 5

15. $\lim\limits_{x\to-\infty}\dfrac{\sqrt{x^2+4x}}{4x+1}=\lim\limits_{x\to-\infty}\dfrac{-\sqrt{1+4/x}}{4+1/x}=\dfrac{-\sqrt{1+0}}{4+0}$

$=-\dfrac{1}{4}$

Note: In dividing numerator and denominator by x, we used the fact that for $x<0$, $x=-\sqrt{x^2}$.

16. $\lim\limits_{x\to\infty}\dfrac{1-\sqrt{x}}{1+\sqrt{x}}=\lim\limits_{x\to\infty}\dfrac{(1/\sqrt{x})-1}{(1/\sqrt{x})+1}=\dfrac{0-1}{0+1}=-1$

17. $\lim\limits_{x\to\infty}\left(\sqrt{x^2+3x+1}-x\right)$

$$=\lim_{x\to\infty}\left(\sqrt{x^2+3x+1}-x\right)\frac{\sqrt{x^2+3x+1}+x}{\sqrt{x^2+3x+1}+x}$$
$$=\lim_{x\to\infty}\frac{x^2+3x+1-x^2}{\sqrt{x^2+3x+1}+x}$$
$$=\lim_{x\to\infty}\frac{3x+1}{\sqrt{x^2+3x+1}+x}$$
$$=\lim_{x\to\infty}\frac{3+1/x}{\sqrt{1+(3/x)+(1/x^2)}+1}$$
$$=\frac{3+0}{\sqrt{1+3\cdot 0+0}+1}=\frac{3}{2}$$

18. $\lim\limits_{x\to\infty}\left(\sqrt{x^2+1}-\sqrt{x^2-1}\right)$

$$=\lim_{x\to\infty}\left(\sqrt{x^2+1}-\sqrt{x^2-1}\right)\frac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}+\sqrt{x^2-1}}$$
$$=\lim_{x\to\infty}\frac{(x^2+1)-(x^2-1)}{\sqrt{x^2+1}+\sqrt{x^2-1}}$$
$$=\lim_{x\to\infty}\frac{2}{\sqrt{x^2+1}+\sqrt{x^2-1}}$$
$$=\lim_{x\to\infty}\frac{2/x}{\sqrt{1+(1/x^2)}+\sqrt{1-(1/x^2)}}$$
$$=\frac{0}{\sqrt{1+0}+\sqrt{1-0}}=0$$

19. $\lim\limits_{x\to\infty}\left(\sqrt{1+x}-\sqrt{x}\right)$

$$=\lim_{x\to\infty}\left(\sqrt{1+x}-\sqrt{x}\right)\left(\frac{\sqrt{1+x}+\sqrt{x}}{\sqrt{1+x}+\sqrt{x}}\right)$$
$$=\lim_{x\to\infty}\frac{(1+x)-x}{\sqrt{1+x}+\sqrt{x}}=\lim_{x\to\infty}\frac{1}{\sqrt{1+x}+\sqrt{x}}$$
$$=\lim_{x\to\infty}\frac{1/\sqrt{x}}{\sqrt{(1/x)+1}+1}=\frac{0}{\sqrt{0+1}+1}=0$$

20. Using $a^3-b^3=(a-b)\left(a^2+ab+b^2\right)$ with $a=\sqrt[3]{1+x}$ and $b=\sqrt[3]{x}$, we have

$\lim\limits_{x\to\infty}\left(\sqrt[3]{1+x}-\sqrt[3]{x}\right)$

$$=\lim_{x\to\infty}\frac{(1+x)-x}{(1+x)^{2/3}+(1+x)^{1/3}x^{1/3}+x^{2/3}}$$
$$=\lim_{x\to\infty}\frac{1}{(1+x)^{2/3}+(1+x)^{1/3}x^{1/3}+x^{2/3}}=0$$

21. $\lim\limits_{x\to-\infty}\left(\sqrt{x^2+x+1}+x\right)$

$$=\lim_{x\to-\infty}\left(\sqrt{x^2+x+1}+x\right)\left[\frac{\sqrt{x^2+x+1}-x}{\sqrt{x^2+x+1}-x}\right]$$
$$=\lim_{x\to-\infty}\frac{x+1}{\left(\sqrt{x^2+x+1}-x\right)}$$
$$=\lim_{x\to-\infty}\frac{1+(1/x)}{-\sqrt{1+(1/x)+(1/x^2)}-1}$$
$$=\frac{1+0}{-\sqrt{1+0+0}-1}=-\frac{1}{2}$$

22. $\lim\limits_{x\to\infty}\left(x+\sqrt{x}\right)=\infty$ since $x\to\infty$ and $\sqrt{x}\to\infty$.

23. $\lim\limits_{x\to-\infty}\left(x^3-5x^2\right)=-\infty$ since $x^3\to-\infty$ and $-5x^2\to-\infty$ as $x\to-\infty$.

Or: $\lim\limits_{x\to-\infty}\left(x^3-5x^2\right)=\lim\limits_{x\to-\infty}x^2\,(x-5)=-\infty$ since $x^2\to\infty$ and $x-5\to-\infty$.

24. $\lim\limits_{x\to\infty}\dfrac{x^7-1}{x^6+1}=\lim\limits_{x\to\infty}\dfrac{1-1/x^7}{(1/x)+(1/x^7)}=\infty$ since $1-\dfrac{1}{x^7}\to 1$ while $\dfrac{1}{x}+\dfrac{1}{x^7}\to 0^+$ as $x\to\infty$.

Or: Divide numerator and denominator by x^6 instead of x^7.

25. As $x\to\infty$, $x^2\to\infty$ and $-x^2\to-\infty$. Thus, $\lim\limits_{x\to\infty}e^{-x^2}=\lim\limits_{t\to-\infty}e^t=0$.

26. $\lim\limits_{x\to\infty}\left(x^2-x^4\right)=\lim\limits_{x\to\infty}x^2\left(1-x^2\right)=-\infty$ since $x^2\to\infty$ and $1-x^2\to-\infty$.

27. $\lim\limits_{x\to\infty}\dfrac{x^3-1}{x^4+1}=\lim\limits_{x\to\infty}\dfrac{(1/x)-(1/x^4)}{1+(1/x^4)}=\dfrac{0-0}{1+0}=0$

28. $\lim\limits_{x\to\infty}\dfrac{\sqrt{x}+3}{x+3}=\lim\limits_{x\to\infty}\dfrac{(1/\sqrt{x})+(3/x)}{1+3/x}=\dfrac{0+0}{1+0}=0$

29. $\lim\limits_{x\to\infty}\dfrac{x}{\sqrt{x-1}}=\lim\limits_{x\to\infty}\dfrac{x/\sqrt{x}}{\sqrt{x-1}/\sqrt{x}}$

$$=\lim_{x\to\infty}\frac{\sqrt{x}}{\sqrt{1-1/x}}=\infty$$

since $\sqrt{x}\to\infty$ and $\sqrt{1-1/x}\to 1$.

Or: Divide numerator and denominator by x instead of $\sqrt{x}$.

30. If $f(x)=x^2\sin\left(5/x^2\right)$, then a calculator gives the following approximate values: $f(1)=-0.95892$, $f(2)=3.79594$, $f(3)=4.74674$, $f(4)=4.91902$, $f(5)=4.96673$, $f(6)=4.98394$, $f(7)=4.99133$, $f(8)=4.99492$, $f(9)=4.99683$, $f(10)=4.99792$, $f(20)=4.99987$, $f(50)=4.999997$, $f(100)=4.9999998$. It appears that $\lim\limits_{x\to\infty}x^2\sin\left(5/x^2\right)=5$.

Proof: Let $t=\dfrac{1}{x^2}$. Then as $x\to\infty$, $t\to 0$ and

$$\lim_{x\to\infty}x^2\sin\frac{5}{x^2}=\lim_{t\to 0}\left(\frac{1}{t}\sin 5t\right)=5\lim_{t\to 0}\frac{\sin 5t}{5t}=5.$$