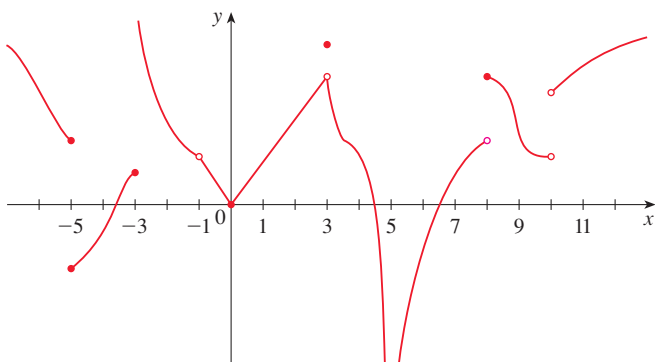


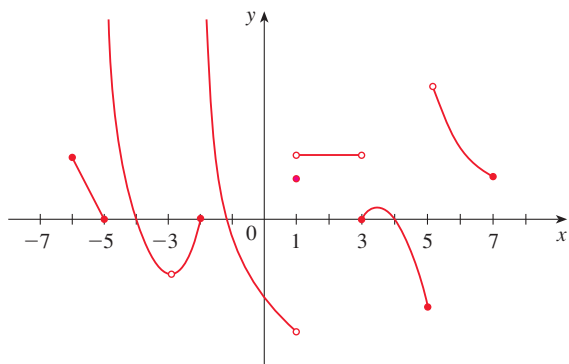
2.4 Continuity

A Click here for answers.

1. (a) From the graph of f , state the numbers at which f is discontinuous and explain why.
 (b) For each of the numbers stated in part (a), determine whether f is continuous from the right, or from the left, or neither.



2. From the graph of g , state the intervals on which g is continuous.



3–6 Use the definition of continuity and the properties of limits to show that the function is continuous at the given number.

3. $f(x) = x^4 - 5x^3 + 6$, $a = 3$

4. $f(x) = x^2 + (x - 1)^9$, $a = 2$

5. $f(x) = 1 + \sqrt{x^2 - 9}$, $a = 5$

6. $g(t) = \frac{\sqrt[3]{t}}{(t + 1)^4}$, $a = -8$

7–10 Use the definition of continuity and the properties of limits to show the function is continuous on the given interval.

7. $f(x) = x + \sqrt{x - 1}$, $[1, \infty)$

8. $f(x) = (x^2 - 1)^8$, $(-\infty, \infty)$

9. $f(x) = x\sqrt{16 - x^2}$, $[-4, 4]$

10. $F(x) = \frac{x + 1}{x - 3}$, $(-\infty, 3)$

S Click here for solutions.

11–16 Explain why the function is discontinuous at the given number a . Sketch the graph of the function.

11. $f(x) = -\frac{1}{(x - 1)^2}$ $a = 1$

12. $f(x) = \begin{cases} -\frac{1}{(x - 1)^2} & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases}$ $a = 1$

13. $f(x) = \frac{x^2 - 1}{x + 1}$ $a = -1$

14. $f(x) = \begin{cases} \frac{x^2 - 1}{x + 1} & \text{if } x \neq -1 \\ 6 & \text{if } x = -1 \end{cases}$ $a = -1$

15. $f(x) = \begin{cases} \frac{x^2 - 2x - 8}{x - 4} & \text{if } x \neq 4 \\ 3 & \text{if } x = 4 \end{cases}$ $a = 4$

16. $f(x) = \begin{cases} 1 - x & \text{if } x \leq 2 \\ x^2 - 2x & \text{if } x > 2 \end{cases}$ $a = 2$

17–25 Use Theorems 4, 5, and 9 to show that the function is continuous on its domain. State the domain.

17. $f(x) = (x + 1)(x^3 + 8x + 9)$

18. $G(x) = \frac{x^4 + 17}{6x^2 + x - 1}$

19. $H(x) = \frac{1}{\sqrt{x + 1}}$

20. $f(t) = 2t + \sqrt{25 - t^2}$

21. $h(x) = \sqrt[3]{x - 1}(x^2 - 2)$

22. $g(t) = \frac{1}{t + \sqrt{t^2 - 4}}$

23. $F(t) = (t^2 + t + 1)^{3/2}$

24. $H(x) = \sqrt{\frac{x - 2}{5 + x}}$

25. $L(x) = |x^3 - x|$

26. Let

$$f(x) = \begin{cases} x - 1 & \text{for } x < 3 \\ 5 - x & \text{for } x \geq 3 \end{cases}$$

Show that f is continuous on $(-\infty, \infty)$.

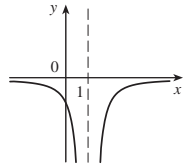
27–31 Find the numbers at which f is discontinuous. At which of these numbers is f continuous from the right, from the left, or neither? Sketch the graph of f .

27. $f(x) = \begin{cases} 2x + 1 & \text{if } x \leq -1 \\ 3x & \text{if } -1 < x < 1 \\ 2x - 1 & \text{if } x \geq 1 \end{cases}$

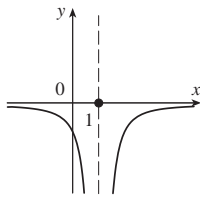
Answers

E Click here for exercises.

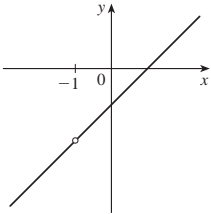
1. (a) -5 (jump), -3 (infinite), -1 (undefined), 3 (removable), 5 (infinite), 8 (jump), 10 (undefined)
 (b) -5 , left; -3 , left; -1 , neither; 3 , neither; 5 , neither; 8 , right; 10 , neither
 2. $[-6, -5]$, $(-5, -3)$, $(-3, -2]$, $(-2, 1)$, $(1, 3)$, $[3, 5]$, $(5, 7]$
 11. $f(1)$ undefined



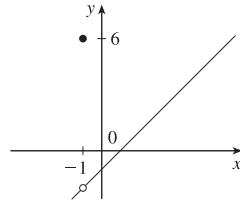
12. $\lim_{x \rightarrow 1} f(x)$ does not exist



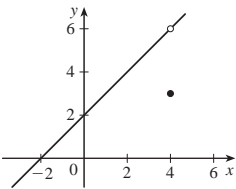
13. $f(-1)$ is not defined



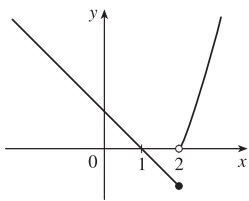
14. $\lim_{x \rightarrow -1} f(x) \neq f(-1)$



15. $\lim_{x \rightarrow 4} f(x) \neq f(4)$



16. $\lim_{x \rightarrow 2} f(x)$ does not exist

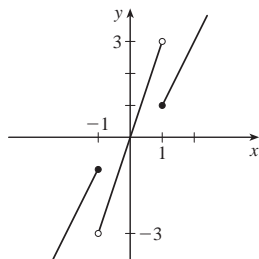


17. \mathbb{R} 18. $\{x \mid x \neq -\frac{1}{2}, \frac{1}{3}\}$ 19. $(-1, \infty)$ 20. $[-5, 5]$

21. \mathbb{R} 22. $(-\infty, -2] \cup [2, \infty)$ 23. \mathbb{R}

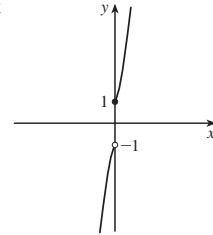
24. $(-\infty, -5) \cup [2, \infty)$ 25. \mathbb{R}

27. -1 , continuous from the left;
 1 , continuous from the right

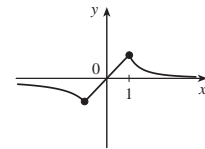


S Click here for solutions.

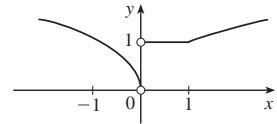
28. 0 , continuous from the right



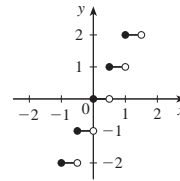
29. Continuous at all points



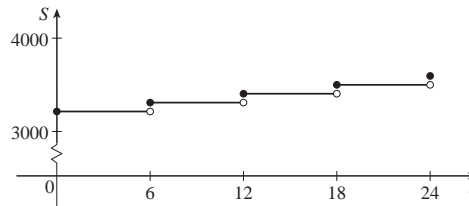
30. 0 , neither



31. $\{n/2 \mid n \text{ an integer}\}$,
 continuous from the right



32. Discontinuous at $t = 6, 12, 8, 24$;
 continuous from the right at $t = 6, 12, 18$



33. $c = 2, d = 0$

Solutions

E Click here for exercises.

1. (a) The following are the numbers at which f is discontinuous and the type of discontinuity at that number: -5 (jump), -3 (infinite), -1 (undefined), 3 (removable), 5 (infinite), 8 (jump), 10 (undefined).
 (b) f is continuous from the left at -5 and -3 , and continuous from the right at 8 . It is continuous from neither side at -1 , 3 , 5 , and 10 .
2. g is continuous on $[-6, -5]$, $(-5, -3)$, $(-3, -2]$, $(-2, 1)$, $(1, 3)$, $[3, 5]$, and $(5, 7]$.
3. $\lim_{x \rightarrow 3} (x^4 - 5x^3 + 6) = \lim_{x \rightarrow 3} x^4 - 5 \lim_{x \rightarrow 3} x^3 + \lim_{x \rightarrow 3} 6$
 $= 3^4 - 5(3^3) + 6 = -48 = f(3)$

Thus f is continuous at 3 .

4. $\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} [x^2 + (x-1)^9]$
 $= \lim_{x \rightarrow 2} x^2 + \left(\lim_{x \rightarrow 2} x - \lim_{x \rightarrow 2} 1 \right)^9$
 $= 2^2 + (2-1)^9 = 5 = f(2)$

Thus f is continuous at 2 .

5. $\lim_{x \rightarrow 5} f(x) = \lim_{x \rightarrow 5} (1 + \sqrt{x^2 - 9})$
 $= \lim_{x \rightarrow 5} 1 + \sqrt{\lim_{x \rightarrow 5} x^2 - \lim_{x \rightarrow 5} 9}$
 $= 1 + \sqrt{5^2 - 9} = 5 = f(5)$

Thus f is continuous at 5 .

6. $\lim_{t \rightarrow -8} g(t) = \lim_{t \rightarrow -8} \frac{\sqrt[3]{t}}{(t+1)^4} = \frac{\sqrt[3]{\lim_{t \rightarrow -8} t}}{\left(\lim_{t \rightarrow -8} t + 1 \right)^4}$
 $= \frac{\sqrt[3]{-8}}{(-8+1)^4} = -\frac{2}{2401} = g(-8)$

Thus g is continuous at -8 .

7. For $a > 1$ we have

$$\begin{aligned} \lim_{x \rightarrow a} f(x) &= \lim_{x \rightarrow a} (x + \sqrt{x-1}) \\ &= \lim_{x \rightarrow a} x + \sqrt{\lim_{x \rightarrow a} x - \lim_{x \rightarrow a} 1} \\ &= a + \sqrt{a-1} = f(a) \end{aligned}$$

so f is continuous on $(1, \infty)$. A similar calculation shows that $\lim_{x \rightarrow 1^+} f(x) = 1 = f(1)$, so f is continuous from the right at 1 . Thus f is continuous on $[1, \infty)$.

8. For any $a \in \mathbb{R}$ we have

$$\begin{aligned} \lim_{x \rightarrow a} f(x) &= \lim_{x \rightarrow a} (x^2 - 1)^8 = \left(\lim_{x \rightarrow a} x^2 - \lim_{x \rightarrow a} 1 \right)^8 \\ &= (a^2 - 1)^8 = f(a) \end{aligned}$$

Thus f is continuous on $(-\infty, \infty)$.

A Click here for answers.

9. For $-4 < a < 4$ we have

$$\begin{aligned} \lim_{x \rightarrow a} f(x) &= \lim_{x \rightarrow a} x \sqrt{16 - x^2} \\ &= \lim_{x \rightarrow a} x \sqrt{\lim_{x \rightarrow a} 16 - \lim_{x \rightarrow a} x^2} \\ &= a \sqrt{16 - a^2} = f(a) \end{aligned}$$

so f is continuous on $(-4, 4)$. Similarly, we get

$$\lim_{x \rightarrow 4^-} f(x) = 0 = f(4) \quad \text{and} \quad \lim_{x \rightarrow -4^+} f(x) = 0 = f(-4),$$

so f is continuous from the left at 4 and from the right at -4 .

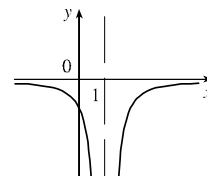
Thus, f is continuous on $[-4, 4]$.

10. For $a < 3$,

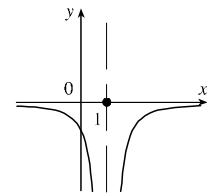
$$\begin{aligned} \lim_{x \rightarrow a} F(x) &= \lim_{x \rightarrow a} \frac{x+1}{x-3} = \frac{\lim_{x \rightarrow a} x + \lim_{x \rightarrow a} 1}{\lim_{x \rightarrow a} x - \lim_{x \rightarrow a} 3} \\ &= \frac{a+1}{a-3} = F(a) \end{aligned}$$

so F is continuous on $(-\infty, 3)$.

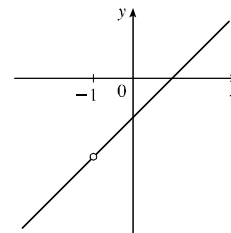
11. $f(x) = -\frac{1}{(x-1)^2}$ is discontinuous at 1 since $f(1)$ is not defined.



12. $\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} \left[-\frac{1}{(x-1)^2} \right]$ does not exist. Therefore f is discontinuous at 1 .

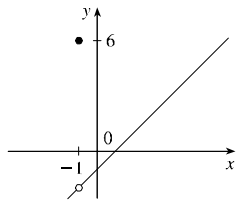


13. $f(x) = \frac{x^2 - 1}{x + 1}$ is discontinuous at -1 because $f(-1)$ is not defined.



14. Since $f(x) = \frac{x^2 - 1}{x + 1}$ for $x \neq -1$, we have

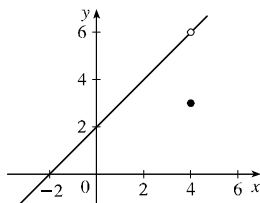
$\lim_{x \rightarrow -1} f(x) = \lim_{x \rightarrow -1} \frac{x^2 - 1}{x + 1} = \lim_{x \rightarrow -1} (x - 1) = -2$. But $f(-1) = 6$, so $\lim_{x \rightarrow -1} f(x) \neq f(-1)$. Therefore, f is discontinuous at -1 .



15. Since $f(x) = \frac{x^2 - 2x - 8}{x - 4}$ if $x \neq 4$,

$$\begin{aligned} \lim_{x \rightarrow 4} f(x) &= \lim_{x \rightarrow 4} \frac{x^2 - 2x - 8}{x - 4} \\ &= \lim_{x \rightarrow 4} \frac{(x - 4)(x + 2)}{x - 4} = \lim_{x \rightarrow 4} (x + 2) \\ &= 4 + 2 = 6 \end{aligned}$$

But $f(4) = 3$, so $\lim_{x \rightarrow 4} f(x) \neq f(4)$. Therefore, f is discontinuous at 4.

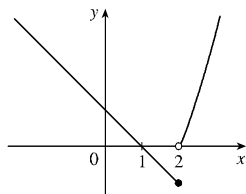


16. $\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (1 - x) = 1 - 2 = -1$ and

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (x^2 - 2x) = (2)^2 - 2(2) = 0. \text{ Since}$$

$$\lim_{x \rightarrow 2^-} f(x) \neq \lim_{x \rightarrow 2^+} f(x), \lim_{x \rightarrow 2} f(x) \text{ does not exist and}$$

therefore f is discontinuous at 2 [by Note 2 after Definition 1].



17. $f(x) = (x + 1)(x^3 + 8x + 9)$ is a polynomial, so by Theorem 5 it is continuous on \mathbb{R} .

18. $G(x) = \frac{x^4 + 17}{6x^2 + x - 1}$ is a rational function, so by Theorem 5 it is continuous on its domain, which is $\{x \mid (3x - 1)(2x + 1) \neq 0\} = \{x \mid x \neq -\frac{1}{2}, \frac{1}{3}\}$.

19. $g(x) = x + 1$, a polynomial, is continuous (by Theorem 5) and $f(x) = \sqrt{x}$ is continuous on $[0, \infty)$ by Theorem 9, so $f(g(x)) = \sqrt{x + 1}$ is continuous on $[-1, \infty)$ by Theorem 9. By Theorem 4 #5, $H(x) = 1/\sqrt{x + 1}$ is continuous on $(-1, \infty)$.

20. $G(t) = 25 - t^2$ is a polynomial, so it is continuous (Theorem 5). $F(x) = \sqrt{x}$ is continuous by Theorem 9. So, by Theorem 9, $F(G(t)) = \sqrt{25 - t^2}$ is continuous on its domain, which is $\{t \mid 25 - t^2 \geq 0\} = \{t \mid |t| \leq 5\} = [-5, 5]$. Also, $2t$ is continuous on \mathbb{R} , so by Theorem 4 #1, $f(t) = 2t + \sqrt{25 - t^2}$ is continuous on its domain, which is $[-5, 5]$.

21. $g(x) = x - 1$ and $G(x) = x^2 - 2$ are both polynomials, so by Theorem 5 they are continuous. Also $f(x) = \sqrt[5]{x}$ is continuous by Theorem 9, so $f(g(x)) = \sqrt[5]{x - 1}$ is continuous on \mathbb{R} . Thus the product $h(x) = \sqrt[5]{x - 1}(x^2 - 2)$ is continuous on \mathbb{R} by Theorem 4 #4.

22. $G(t) = t^2 - 4$ is continuous since it is a polynomial (Theorem 5). $F(x) = \sqrt{x}$ is continuous by Theorem 7. So, by Theorem 9, $F(G(t)) = \sqrt{t^2 - 4}$ is continuous on its domain, which is $D = \{t \mid t^2 - 4 \geq 0\} = \{t \mid |t| \geq 2\}$. Also t is continuous so $t + \sqrt{t^2 - 4}$ is continuous on D by Theorem 4 #1. Thus by Theorem 4 #5, $g(t) = 1/(t + \sqrt{t^2 - 4})$ is continuous on its domain, which is $\{t \in D \mid t + \sqrt{t^2 - 4} \neq 0\}$. But if $t + \sqrt{t^2 - 4} = 0$, then $\sqrt{t^2 - 4} = -t \Rightarrow t^2 - 4 = t^2 \Rightarrow -4 = 0$ which is false. So the domain of g is $\{t \in D \mid |t| \geq 2\} = (-\infty, -2] \cup [2, \infty)$.

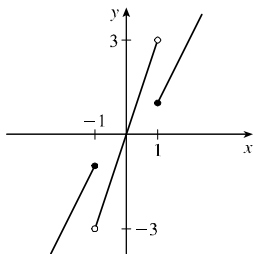
23. Since the discriminant of $t^2 + t + 1$ is negative, $t^2 + t + 1$ is always positive. So the domain of $F(t)$ is \mathbb{R} . By Theorem 5 the polynomial $(t^2 + t + 1)^3$ is continuous. By Theorems 6 and 9 the composition $F(t) = \sqrt{(t^2 + t + 1)^3}$ is continuous on \mathbb{R} .

24. $H(x) = \sqrt{(x - 2)/(5 + x)}$. The domain is $\{x \mid (x - 2)/(5 + x) > 0\} = (-\infty, -5) \cup [2, \infty)$ by the methods of Appendix A. By Theorem 5 the rational function $(x - 2)/(5 + x)$ is continuous. Since the square root function is continuous (Theorem 6), the composition $H(x) = \sqrt{(x - 2)/(5 + x)}$ is continuous on its domain by Theorem 9.

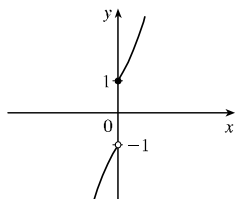
25. $g(x) = x^3 - x$ is continuous on \mathbb{R} since it is a polynomial [Theorem 5(a)], and $f(x) = |x|$ is continuous on \mathbb{R} . So $L(x) = |x^3 - x|$ is continuous on \mathbb{R} by Theorem 9.

26. f is continuous on $(-\infty, 3)$ and $(3, \infty)$ since on each of these intervals it is a polynomial. Also $\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} (5 - x) = 2$ and $\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} (x - 1) = 2$, so $\lim_{x \rightarrow 3} f(x) = 2$. Since $f(3) = 5 - 3 = 2$, f is also continuous at 3. Thus, f is continuous on $(-\infty, \infty)$.

27. f is continuous on $(-\infty, -1)$, $(-1, 1)$ and $(1, \infty)$ since on each of these intervals it is a polynomial. Now $\lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^-} (2x + 1) = -1$ and $\lim_{x \rightarrow -1^+} f(x) = \lim_{x \rightarrow -1^+} 3x = -3$, so f is discontinuous at -1 . Since $f(-1) = -1$, f is continuous from the left at -1 . Also $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} 3x = 3$ and $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (2x - 1) = 1$, so f is discontinuous at 1. Since $f(1) = 1$, f is continuous from the right at 1.

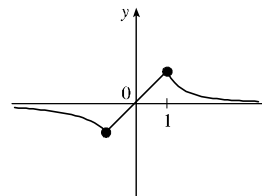


28. f is continuous on $(-\infty, 0)$ and $(0, \infty)$ since on each of these intervals it is a polynomial. Now $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (x - 1)^3 = -1$ and $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (x + 1)^3 = 1$. Thus, $\lim_{x \rightarrow 0} f(x)$ does not exist, so f is discontinuous at 0. Since $f(0) = 1$, f is continuous from the right at 0.

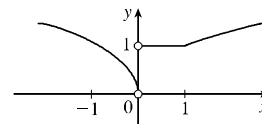


29. f is continuous on $(-\infty, -1)$, $(-1, 1)$ and $(1, \infty)$.

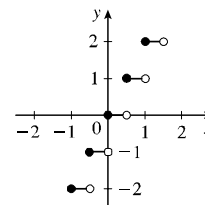
Now $\lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^-} \frac{1}{x} = -1$ and $\lim_{x \rightarrow -1^+} f(x) = \lim_{x \rightarrow -1^+} x = -1$, so $\lim_{x \rightarrow -1} f(x) = -1 = f(-1)$ and f is continuous at -1 . Also $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} x = 1$ and $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \frac{1}{x^2} = 1$, so $\lim_{x \rightarrow 1} f(x) = 1 = f(1)$ and f is continuous at 1. Thus f has no discontinuities.



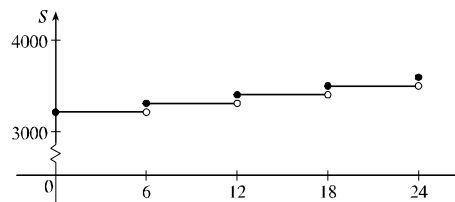
30. f is continuous on $(-\infty, 0)$, $(0, 1)$ and $(1, \infty)$. Since f is not defined at $x = 0$, f is continuous neither from the right nor the left at 0. Also $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} 1 = 1$ and $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \sqrt{x} = 1$, so $\lim_{x \rightarrow 1} f(x) = 1 = f(1)$ and f is continuous at 1.



31. $f(x) = \llbracket 2x \rrbracket$ is continuous except when $2x = n \Leftrightarrow x = n/2$, n an integer. In fact, $\lim_{x \rightarrow n/2^-} \llbracket 2x \rrbracket = n - 1$ and $\lim_{x \rightarrow n/2^+} \llbracket 2x \rrbracket = n = f(n)$, so f is continuous only from the right at $n/2$.



32. The salary function has discontinuities at $t = 6, 12, 18,$ and 24 , but is continuous from the right at 6, 12, and 18.



33. The functions $2x$, $cx^2 + d$ and $4x$ are continuous on their own domains, so the only possible problems occur at $x = 1$ and $x = 2$. The left- and right-hand limits at these points must be the same in order for $\lim_{x \rightarrow 1} h(x)$ and $\lim_{x \rightarrow 2} h(x)$ to exist. So we must have $2 \cdot 1 = c(1)^2 + d$ and $c(2)^2 + d = 4 \cdot 2$. From the first of these equations we get $d = 2 - c$. Substituting this into the second, we get $4c + (2 - c) = 8 \Leftrightarrow c = 2$. Back-substituting into the first to get d , we find that $d = 0$.
34. $g(x) = x^5 - 2x^3 + x^2 + 2$ is continuous on $[-2, -1]$ and $g(-2) = -10$, $g(-1) = 4$. Since $-10 < -1 < 4$, there is a number c in $(-2, -1)$ such that $g(c) = -1$ by the Intermediate Value Theorem.
35. $f(x) = x^3 - 3x + 1$ is continuous on $[0, 1]$ and $f(0) = 1$, $f(1) = -1$. Since $-1 < 0 < 1$, there is a number c in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus there is a root of the equation $x^3 - 3x + 1 = 0$ in the interval $(0, 1)$.
36. $f(x) = x^5 - 2x^4 - x - 3$ is continuous on $[2, 3]$ and $f(2) = -5$, $f(3) = 75$. Since $-5 < 0 < 75$, there is a number c in $(2, 3)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus there is a root of the equation $x^5 - 2x^4 - x - 3 = 0$ in the interval $(2, 3)$.
37. $f(x) = x^3 + 2x - (x^2 + 1) = x^3 + 2x - x^2 - 1$ is continuous on $[0, 1]$ and $f(0) = -1$, $f(1) = 1$. Since $-1 < 0 < 1$, there is a number c in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus there is a root of the equation $x^3 + 2x - x^2 - 1 = 0$, or equivalently, $x^3 + 2x = x^2 + 1$, in the interval $(0, 1)$.
38. $f(x) = x^2 - \sqrt{x+1}$ is continuous on $[1, 2]$ and $f(1) = 1 - \sqrt{2}$, $f(2) = 4 - \sqrt{3}$. Since $1 - \sqrt{2} < 0 < 4 - \sqrt{3}$, there is a number c in $(1, 2)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus there is a root of the equation $x^2 - \sqrt{x+1} = 0$, or $x^2 = \sqrt{x+1}$, in the interval $(1, 2)$.