2.3 Calculating Limits Using the Limit Laws

A Click here for answers.

S Click here for solutions.

1–5 Evaluate the limit and justify each step by indicating the appropriate Limit Law(s).

1. $\lim_{x \to 4} (5x^2 - 2x + 3)$

2. $\lim_{x \to 3} (x^3 + 2)(x^2 - 5x)$

3. $\lim_{x \to -1} \frac{x - 2}{x^2 + 4x - 3}$

4. $\lim_{x \to 1} \left(\frac{x^4 + x^2 - 6}{x^4 + 2x + 3} \right)^2$

5. $\lim_{t \to -2} (t + 1)^9 (t^2 - 1)$

6–20 Evaluate the limit, if it exists.

6. $\lim_{x \to -3} \frac{x^2 - x + 12}{x + 3}$

7. $\lim_{x \to -3} \frac{x^2 - x - 12}{x + 3}$

8. $\lim_{x \to -2} \frac{x + 2}{x^2 - x - 6}$

9. $\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - 3x + 2}$

10. $\lim_{h \to 0} \frac{(h - 5)^2 - 25}{h}$

11. $\lim_{x \to -1} \frac{x^2 - x - 2}{x + 1}$

12. $\lim_{x \to 1} \frac{x^2 - x - 2}{x + 1}$

13. $\lim_{t \to 1} \frac{t^3 - t}{t^2 - 1}$

14. $\lim_{x \to -1} \frac{x^2 - x - 3}{x + 1}$

15. $\lim_{t \to 2} \frac{t^2 + t - 6}{t^2 - 4}$

16. $\lim_{t \to 0} \frac{\sqrt{2 - t} - \sqrt{2}}{t}$

17. $\lim_{x \to 1} \left[\frac{1}{x - 1} - \frac{2}{x^2 - 1} \right]$

18. $\lim_{x \to 2} \frac{\frac{1}{x} - \frac{1}{2}}{x - 2}$

19. $\lim_{x \to 0} \frac{x}{\sqrt{1 + 3x} - 1}$

20. $\lim_{x \to 2} \frac{x - \sqrt{3x - 2}}{x^2 - 4}$

21. Prove that $\lim_{x \to 0^+} \sqrt{x} \cos^4 x = 0$.

22. Let

$$f(x) = \begin{cases} x^2 - 2x + 2 & \text{if } x < 1 \\ 3 - x & \text{if } x \geqslant 1 \end{cases}$$

(a) Find $\lim_{x \to 1^-} f(x)$ and $\lim_{x \to 1^+} f(x)$.
(b) Does $\lim_{x \to 1} f(x)$ exist?
(c) Sketch the graph of f.

23. Let

$$g(x) = \begin{cases} -x^3 & \text{if } x < -1 \\ (x + 2)^2 & \text{if } x > -1 \end{cases}$$

(a) Find $\lim_{x \to 1^-} g(x)$ and $\lim_{x \to -1^+} g(x)$.
(b) Does $\lim_{x \to -1} g(x)$ exist?
(c) Sketch the graph of g.

24. Let $g(x) = [\![x/2]\!]$.
(a) Sketch the graph of g.
(b) Evaluate each of the following limits if it exists.

(i) $\lim_{x \to 1^+} g(x)$ (ii) $\lim_{x \to 1^-} g(x)$ (iii) $\lim_{x \to 1} g(x)$

(iv) $\lim_{x \to 2^+} g(x)$ (v) $\lim_{x \to 2^-} g(x)$ (vi) $\lim_{x \to 2} g(x)$

(b) For what values of a does $\lim_{x \to a} g(x)$ exist?

Answers

E Click here for exercises.

S Click here for solutions.

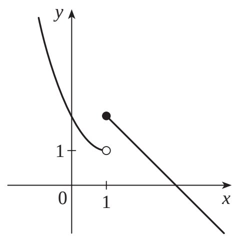
1. 75 **2.** -174 **3.** $\frac{1}{2}$ **4.** $\frac{4}{9}$ **5.** -3

6. Does not exist **7.** -7 **8.** $-\frac{1}{5}$ **9.** -3 **10.** -10

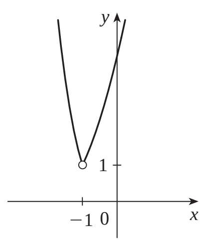
11. -3 **12.** -1 **13.** 1 **14.** Does not exist **15.** $\frac{5}{4}$

16. $-\sqrt{2}/4$ **17.** $\frac{1}{2}$ **18.** $-\frac{1}{4}$ **19.** $\frac{2}{3}$ **20.** $\frac{1}{16}$ **21.** 0

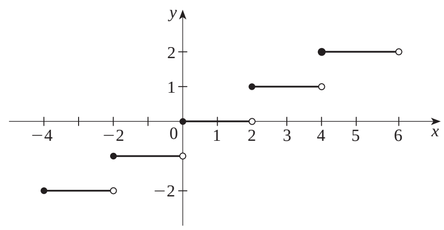
22. (a) 1, 2 (b) No (c)



23. (a) 1, 1 (b) Yes (c)



24. (a)



(b) (i) 0 (ii) 0 (iii) 0 (iv) 1 (v) 0
(vi) Does not exist

(c) All values except even integers

Solutions

E Click here for exercises.

A Click here for answers.

1. $\lim\limits_{x\to4}\left(5x^2-2x+3\right)$

$= \lim\limits_{x\to4}5x^2-\lim\limits_{x\to4}2x+\lim\limits_{x\to4}3$ (Limit Laws 2 & 1)

$= 5\lim\limits_{x\to4}x^2-2\lim\limits_{x\to4}x+3$ (3 & 7)

$= 5(4)^2-2(4)+3=75$ (9 & 8)

2. $\lim\limits_{x\to3}\left(x^3+2\right)\left(x^2-5x\right)$

$= \lim\limits_{x\to3}\left(x^3+2\right)\lim\limits_{x\to3}\left(x^2-5x\right)$ (Limit Law 4)

$= \left(\lim\limits_{x\to3}x^3+\lim\limits_{x\to3}2\right)\left(\lim\limits_{x\to3}x^2-5\lim\limits_{x\to3}x\right)$ (1, 2 & 3)

$= \left(3^3+2\right)\left(3^2-5\cdot3\right)$ (9, 7 & 8)

$= 29(-6)=-174$

3. $\lim\limits_{x\to-1}\dfrac{x-2}{x^2+4x-3}$

$= \dfrac{\lim\limits_{x\to-1}(x-2)}{\lim\limits_{x\to-1}\left(x^2+4x-3\right)}$ (Limit Law 5)

$= \dfrac{\lim\limits_{x\to-1}x-\lim\limits_{x\to-1}2}{\lim\limits_{x\to-1}x^2+4\lim\limits_{x\to-1}x-\lim\limits_{x\to-1}3}$ (2, 1 & 3)

$= \dfrac{(-1)-2}{(-1)^2+4(-1)-3}=\dfrac{1}{2}$ (8, 7 & 9)

4. $\lim\limits_{x\to1}\left(\dfrac{x^4+x^2-6}{x^4+2x+3}\right)^2$

$= \left[\dfrac{\lim\limits_{x\to1}\left(x^4+x^2-6\right)}{\lim\limits_{x\to1}\left(x^4+2x+3\right)}\right]^2$ (Limit Laws 6 & 5)

$= \left(\dfrac{\lim\limits_{x\to1}x^4+\lim\limits_{x\to1}x^2-\lim\limits_{x\to1}6}{\lim\limits_{x\to1}x^4+2\lim\limits_{x\to1}x+\lim\limits_{x\to1}3}\right)^2$ (1, 2 & 3)

$= \left(\dfrac{1^4+1^2-6}{1^4+2\cdot1+3}\right)^2$ (9, 7 & 8)

$= \left(\frac{-4}{6}\right)^2=\left(-\frac{2}{3}\right)^2=\frac{4}{9}$

5. $\lim\limits_{t\to-2}(t+1)^9\left(t^2-1\right)$

$= \lim\limits_{t\to-2}(t+1)^9\lim\limits_{t\to-2}\left(t^2-1\right)$ (Limit Law 4)

$= \left[\lim\limits_{t\to-2}(t+1)\right]^9\lim\limits_{t\to-2}\left(t^2-1\right)$ (6)

$= \left[\lim\limits_{t\to-2}t+\lim\limits_{t\to-2}1\right]^9\left[\lim\limits_{t\to-2}t^2-\lim\limits_{t\to-2}1\right]$ (1 & 2)

$= [(-2)+1]^9\left[(-2)^2-1\right]=-3$ (8, 7 & 9)

6. $\lim\limits_{x\to-3}\dfrac{x^2-x+12}{x+3}$ does not exist since $x+3\to0$ but $x^2-x+12\to24$ as $x\to-3$.

7. $\lim\limits_{x\to-3}\dfrac{x^2-x-12}{x+3}=\lim\limits_{x\to-3}\dfrac{(x+3)(x-4)}{x+3}$

$=\lim\limits_{x\to-3}(x-4)=-3-4=-7$

8. $\lim\limits_{x\to-2}\dfrac{x+2}{x^2-x-6}=\lim\limits_{x\to-2}\dfrac{x+2}{(x-3)(x+2)}$

$=\lim\limits_{x\to-2}\dfrac{1}{x-3}=-\dfrac{1}{5}$

9. $\lim\limits_{x\to1}\dfrac{x^2+x-2}{x^2-3x+2}=\lim\limits_{x\to1}\dfrac{(x+2)(x-1)}{(x-2)(x-1)}$

$=\lim\limits_{x\to1}\dfrac{x+2}{x-2}=\dfrac{1+2}{1-2}=-3$

10. $\lim\limits_{h\to0}\dfrac{(h-5)^2-25}{h}=\lim\limits_{h\to0}\dfrac{\left(h^2-10h+25\right)-25}{h}$

$=\lim\limits_{h\to0}\dfrac{h^2-10h}{h}=\lim\limits_{h\to0}(h-10)$

$=-10$

11. $\lim\limits_{x\to-1}\dfrac{x^2-x-2}{x+1}=\lim\limits_{x\to-1}\dfrac{(x+1)(x-2)}{x+1}$

$=\lim\limits_{x\to-1}(x-2)=-3$

12. $\lim\limits_{x\to1}\dfrac{x^2-x-2}{x+1}=\dfrac{1^2-1-2}{1+1}=-1$

13. $\lim\limits_{t\to1}\dfrac{t^3-t}{t^2-1}=\lim\limits_{t\to1}\dfrac{t\left(t^2-1\right)}{t^2-1}=\lim\limits_{t\to1}t=1$

14. $\lim\limits_{x\to-1}\dfrac{x^2-x-3}{x+1}$ does not exist since as $x\to-1$, numerator $\to-1$ and denominator $\to0$.

15. $\lim\limits_{t\to2}\dfrac{t^2+t-6}{t^2-4}=\lim\limits_{t\to2}\dfrac{(t+3)(t-2)}{(t+2)(t-2)}=\lim\limits_{t\to2}\dfrac{t+3}{t+2}=\dfrac{5}{4}$

16. $\lim\limits_{t\to0}\dfrac{\sqrt{2-t}-\sqrt{2}}{t}=\lim\limits_{t\to0}\dfrac{\sqrt{2-t}-\sqrt{2}}{t}\cdot\dfrac{\sqrt{2-t}+\sqrt{2}}{\sqrt{2-t}+\sqrt{2}}$

$=\lim\limits_{t\to0}\dfrac{-t}{t\left(\sqrt{2-t}+\sqrt{2}\right)}$

$=\lim\limits_{t\to0}\dfrac{-1}{\sqrt{2-t}+\sqrt{2}}=-\dfrac{1}{2\sqrt{2}}$

$=-\dfrac{\sqrt{2}}{4}$

17. $\lim\limits_{x\to1}\left(\dfrac{1}{x-1}-\dfrac{2}{x^2-1}\right)=\lim\limits_{x\to1}\dfrac{(x+1)-2}{(x-1)(x+1)}$

$=\lim\limits_{x\to1}\dfrac{x-1}{(x-1)(x+1)}$

$=\lim\limits_{x\to1}\dfrac{1}{x+1}=\dfrac{1}{2}$

18. $\lim\limits_{x\to2}\dfrac{1/x-\frac{1}{2}}{x-2}=\lim\limits_{x\to2}\dfrac{2-x}{2x(x-2)}=\lim\limits_{x\to2}\dfrac{-1}{2x}=-\dfrac{1}{4}$

19. $$\lim_{x\to 0}\frac{x}{\sqrt{1+3x}-1}=\lim_{x\to 0}\frac{x\left(\sqrt{1+3x}+1\right)}{\left(\sqrt{1+3x}-1\right)\left(\sqrt{1+3x}+1\right)}=\lim_{x\to 0}\frac{x\left(\sqrt{1+3x}+1\right)}{3x}=\lim_{x\to 0}\frac{\sqrt{1+3x}+1}{3}=\frac{\sqrt{1}+1}{3}=\frac{2}{3}$$

20. $$\lim_{x\to 2}\frac{x-\sqrt{3x-2}}{x^2-4}=\lim_{x\to 2}\frac{\left(x-\sqrt{3x-2}\right)\left(x-\sqrt{3x-2}\right)}{\left(x^2-4\right)\left(x-\sqrt{3x-2}\right)}=\lim_{x\to 2}\frac{x^2-3x+2}{\left(x^2-4\right)\left(x+\sqrt{3x-2}\right)}=\lim_{x\to 2}\frac{(x-2)(x-1)}{(x-2)(x+2)\left(x+\sqrt{3x-2}\right)}=\lim_{x\to 2}\frac{(x-1)}{(x+2)\left(x+\sqrt{3x-2}\right)}=\frac{1}{4\left(2+\sqrt{4}\right)}=\frac{1}{16}$$

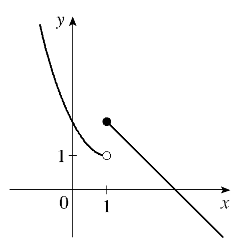
21. $1\le\cos x\le 1 \;\Rightarrow\; 0\le\cos^4 x\le 1 \;\Rightarrow$ $0\le\sqrt{x}\cos^4 x\le\sqrt{x}$. But $\lim_{x\to 0^+}0=\lim_{x\to 0^+}\sqrt{x}=0$. So by the Squeeze Theorem, $\lim_{x\to 0^+}\sqrt{x}\cos^4 x=0$.

22. (a) $$\lim_{x\to 1^-}f(x)=\lim_{x\to 1^-}\left(x^2-2x+2\right)=\lim_{x\to 1^-}x^2-2\lim_{x\to 1^-}x+\lim_{x\to 1^-}2=1^2-2+2=1$$,

$$\lim_{x\to 1^+}f(x)=\lim_{x\to 1^+}(3-x)=\lim_{x\to 1^+}3-\lim_{x\to 1^+}x=3-1=2$$

(b) $\lim_{x\to 1}f(x)$ does not exist because $\lim_{x\to 1^-}f(x)\ne\lim_{x\to 1^+}f(x)$.

(c)

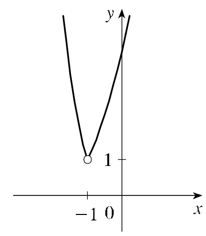


23. (a) $$\lim_{x\to -1^-}g(x)=\lim_{x\to -1^-}\left(-x^3\right)=-(-1)^3=1,$$

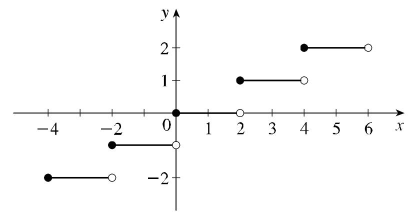
$$\lim_{x\to -1^+}g(x)=\lim_{x\to -1^+}(x+2)^2=(-1+2)^2=1$$

(b) By part (a), $\lim_{x\to -1}g(x)=1$.

(c)



24. (a)



(b) (i) $\lim_{x\to 1^+}g(x)=0$ since $[\![x/2]\!]=0$ for $0\le x<2$.

(ii) $\lim_{x\to 1^-}g(x)=0$ since $[\![x/2]\!]=0$ for $0\le x<2$.

(iii) $\lim_{x\to 1}g(x)=0$ since $[\![x/2]\!]=0$ for $0\le x<2$.

(iv) $\lim_{x\to 2^+}g(x)=1$ since $[\![x/2]\!]=1$ for $2\le x<4$.

(v) $\lim_{x\to 2^-}g(x)=0$ since $[\![x/2]\!]=0$ for $0\le x<2$.

(vi) $\lim_{x\to 2}g(x)$ does not exist because $\lim_{x\to 2^+}g(x)\ne\lim_{x\to 2^-}g(x)$.

(c) $\lim_{x\to a}g(x)$ exists except when a is an even integer.